Asymptotic moments of spatial branching processes

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(P, G)-BRANCHING MARKOV PROCESS

- Particles will live in E a Lusin space (e.g. a Polish space would be enough)
- Let $P = (P_t, t \geq 0)$ be a semigroup on E.
- Write $B^+(E)$ for non-negative bounded measurable functions on E

- Particles evolve independently according to a P-Markov process.
- In an event which we refer to as 'branching', particles positioned at x die at rate $\beta \in B^+(E)$ and instantaneously, new particles are created in E according to a point process.
- The configurations of these offspring are described by the random counting measure

$$\mathcal{Z}(A) = \sum_{i=1}^{N} \delta_{x_i}(A),$$

with probabilities $\mathcal{P}_x$, where $x \in E$ is the position of death of the parent.

- Without loss of generality we can assume that $\mathcal{P}_x(N = 1) = 0$. On the other hand, we do allow for the possibility that $\mathcal{P}_x(N = 0) > 0$ for some or all $x \in E$.
- Henceforth we refer to this spatial branching process as a (P, G)-branching Markov process.

(P, G)-BRANCHING MARKOV PROCESS

- Define the so-called branching mechanism

$$G[f](x) := \beta(x)\mathcal{E}_x\left[\prod_{i=1}^{N} f(x_i) - f(x)\right], \qquad x \in E,$$

  where we recall $f \in B_1^+(E) := \{f \in B^+(E) : \sup_{x \in E} f(x) \le 1\}$.

- Configuration of particles at time t is denoted by $\{x_1(t), \ldots, x_{N_t}(t)\}$ and, on the event that the process has not become extinct or exploded,

$$X_t(\cdot) = \sum_{i=1}^{N_t} \delta_{x_i(t)}(\cdot), \qquad t \ge 0.$$

  is Markovian in $N(E)$, the space of integer atomic measures.

- Its probabilities will be denoted $\mathbb{P} := (\mathbb{P}_\mu, \mu \in N(E))$.

- Define,

$$\mathrm{v}_t[f](x) = \mathbb{E}_{\delta_x}\left[\prod_{i=1}^{N_t} f(x_i(t))\right], \qquad f \in B_1^+(E), t \ge 0.$$

- Non-linear evolution semigroup

$$\mathrm{v}_t[f](x) = \hat{\mathrm{P}}_t[f](x) + \int_0^t \mathrm{P}_s\left[G[\mathrm{v}_{t-s}[f]]\right](x)\mathrm{d}s, \qquad t \ge 0.$$

k-TH MOMENT

- Our main results concern understanding the growth of the *k*-th moment functional in time

$$\mathtt{T}_t^{(k)}[f](x) := \mathbb{E}_{\delta_x}[\langle f, X_t \rangle^k], \qquad x \in E, f \in B^+(E), k \geq 1, t \geq 0.$$

- **Notational convenience**: Write $\mathtt{T}_t$ in place of $\mathtt{T}_t^{(1)}$

- **Related historical work**: A number of papers have opened the topic of moments for branching particle systems and superprocesses, including e.g. :

    ○ E. Dumonteil and A. Mazzolo. Residence times of branching diffusion processes.
    Phys. Rev. E, 94:012131, 2016.

    ○ J. Fleischman. Limiting distributions for branching random fields.
    Trans. Amer. Math. Soc., 239:353–389, 1978.

    ○ I. Iscoe. On the supports of measure-valued critical branching Brownian motion.
    Ann. Probab., 16(1):200–221, 1988.

    ○ A. Klenke. Multiple scale analysis of clusters in spatial branching models.
    Ann. Probab., 25(4):1670–1711, 1997.

- Our objective: to show that for $k \geq 2$ and any positive bounded measurable function f on E,

$$\lim_{t \to \infty} g_k(t) \mathbb{E}_{\delta_x}[\langle f, X_t \rangle^k] = C_k(x, f)$$

where the constant $C_k(x, f)$ can be identified explicitly.

- We need **two** fundamental assumptions.

ASSUMPTION (H1): ASMUSSEN-HERING CLASS

There exists an eigenvalue $\lambda \in \mathbb{R}$ and a corresponding right eigenfunction $\varphi \in B^+(E)$ and finite left eigenmeasure $\tilde{\varphi}$ such that, for $f \in B^+(E)$,

$$\langle T_t[\varphi], \mu \rangle = e^{\lambda t} \langle \varphi, \mu \rangle \text{ and } \langle T_t[f], \tilde{\varphi} \rangle = e^{\lambda t} \langle f, \tilde{\varphi} \rangle,$$

for all $\mu \in N(E)$ if $(X, \mathbb{P})$ is a branching Markov process (resp. a superprocess). Further let us define

$$\Delta_t = \sup_{x \in E, f \in B_1^+(E)} |\varphi(x)^{-1} e^{-\lambda t} T_t[f](x) - \langle \tilde{\varphi}, f \rangle|, \qquad t \geq 0.$$

We suppose that

$$\sup_{t \geq 0} \Delta_t < \infty \text{ and } \lim_{t \to \infty} \Delta_t = 0.$$

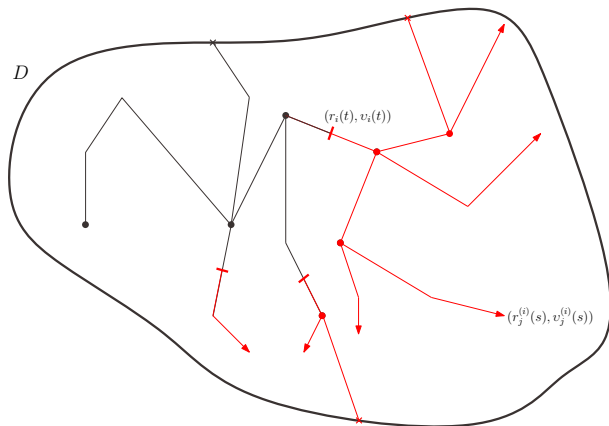
NOTE: This assumption allows us to talk about criticality of the (P, G)-BMP:

$$\lambda = 0 \text{ (critical)} \mid \lambda > 0 \text{ (supercritical)} \mid \lambda < 0 \text{ (subcritical)}$$

WHO LIVES IN THE ASMUSSEN-HERING CLASS?

- Branching Brownian Motion in a bounded domain
- Neutron Branching process in a Bounded domain

ASSUMPTION (H2)$_k$

$$\sup_{x \in E} \mathcal{E}_x(\langle 1, \mathcal{Z} \rangle^k) < \infty.$$

THEOREM: THE CRITICAL CASE ($\lambda = 0$)

Suppose that (H1) holds along with (H2)$_k$ for some $k \geq 2$ and $\lambda = 0$. Define

$$\Delta_t^{(\ell)} = \sup_{x \in E, ||f|| \leq 1} \left| t^{-(\ell-1)} \varphi(x)^{-1} \mathbb{T}_t^{(\ell)}[f](x) - 2^{-(\ell-1)} \ell! \langle f, \tilde{\varphi} \rangle^\ell \langle \mathbb{V}[\varphi], \tilde{\varphi} \rangle^{\ell-1} \right|,$$

where

$$\mathbb{V}[\varphi](x) = \beta(x) \mathcal{E}_x \left(\langle \varphi, \mathcal{Z} \rangle^2 - \langle \varphi^2, \mathcal{Z} \rangle \right).$$

Then, for all $\ell \leq k$

$$\sup_{t \geq 0} \Delta_t^{(\ell)} < \infty \text{ and } \lim_{t \to \infty} \Delta_t^{(\ell)} = 0.$$

In short, subject to (H1) at criticality and (H2)$_k$, we have, for $f \in B_1^+(E)$,

$$\lim_{t \to \infty} t^{-(k-1)} \mathbb{E}_{\delta_x} \left[\langle f, X_t \rangle^k \right] = 2^{-(k-1)} k! \langle f, \tilde{\varphi} \rangle^k \langle \mathbb{V}[\varphi], \tilde{\varphi} \rangle^{k-1} \varphi(x)$$

"At criticality the k-th moment scales like t^{k-1}"

IDEAS FROM THE PROOF

- The obvious starting point:

$$\mathrm{T}_t^{(k)}[f](x) = (-1)^k \frac{\partial^k}{\partial \theta^k} \mathbb{E}_{\delta_x}[\mathrm{e}^{-\theta \langle f, X_t \rangle}]\Bigg|_{\theta=0}$$

- Recall that

$$\mathrm{v}_t[f](x) = \mathbb{E}_{\delta_x}\left[\prod_{i=1}^{N_t} f(x_i(t))\right], \qquad f \in B_1^+(E), t \geq 0.$$

- Non-linear evolution semigroup

$$\mathrm{v}_t[f](x) = \hat{\mathrm{P}}_t[f](x) + \int_0^t \mathrm{P}_s\left[\mathrm{G}[\mathrm{v}_{t-s}[f]]\right](x)\mathrm{d}s, \qquad t \geq 0.$$

- Hence

$$\mathrm{v}_t[\mathrm{e}^{-\theta f}](x) = \mathbb{E}_{\delta_x}[\mathrm{e}^{-\theta \langle f, X_t \rangle}]$$

- We need a new representation of the non-linear semigroup $(\mathrm{v}_t, t \geq 0)$ which connects us to the assumption (H1).

- Recall

$$\mathtt{T}_t[f](x) = \mathtt{T}_t^{(1)}[f](x) = \mathbb{E}_{\delta_x}[\langle f, X_t \rangle], \qquad t \geq 0, f \in B_1^+(E), x \in E.$$

- For $f \in B^+(E)$, it is well known that the mean semigroup evolution satisfies

$$\mathtt{T}_t[f](x) = \mathtt{P}_t[f] + \int_0^t \mathtt{P}_s\left[\mathtt{F}\mathtt{T}_{t-s}[f]\right](x)\mathrm{d}s \qquad t \geq 0, x \in E, \tag{1}$$

where

$$\mathtt{F}[f](x) = \beta(x)\mathcal{E}_x\left[\sum_{i=1}^N f(x_i) - f(x)\right], \qquad x \in E.$$

LINEAR TO NON-LINEAR SEMIGROUP

We now define a variant of the non-linear evolution semigroup equation

$$\mathrm{u}_t[f](x) = \mathbb{E}_{\delta_x}\left[1 - \prod_{i=1}^{N_t} f(x_i(t))\right], \qquad t \geq 0, \, x \in E, f \in B_1^+(E).$$

For $f \in B_1^+(E)$, define

$$\mathrm{A}[f](x) = \beta(x)\mathcal{E}_x\left[\prod_{i=1}^{N}(1 - f(x_i)) - 1 + \sum_{i=1}^{N} f(x_i)\right], \qquad x \in E.$$

$$\mathrm{v}_t[f](x) = \hat{\mathrm{P}}_t[f](x) + \int_0^t \mathrm{P}_s\left[\mathrm{G}[\mathrm{v}_{t-s}[f]]\right](x)\mathrm{d}s \quad \text{and} \quad \mathrm{T}_t[f](x) = \mathrm{P}_t[f] + \int_0^t \mathrm{P}_s\left[\mathrm{FT}_{t-s}[f]\right](x)\mathrm{d}s$$

gives us.....

Lemma
For all $g \in B_1^+(E)$, $x \in E$ and $t \geq 0$, the non-linear semigroup $\mathrm{u}_t[g](x)$ satisfies

$$\mathrm{u}_t[g](x) = \mathrm{T}_t[1 - g](x) - \int_0^t \mathrm{T}_s\left[\mathrm{A}[\mathrm{u}_{t-s}[g]]\right](x)\mathrm{d}s.$$

NONLINEAR TO K-TH MOMENT EVOLUTION EQUATION

In terms of our new semigroup equation:

$$T_t^{(k)}[f](x) = (-1)^{k+1} \frac{\partial^k}{\partial \theta^k} u_t[e^{-\theta f}](x)\bigg|_{\theta=0}.$$

Theorem

Fix $k \geq 2$. Assuming (H1) and (H2)$_k$, with the additional assumption that

$$\sup_{x \in E, s \leq t} T_s^{(\ell)}[f](x) < \infty, \qquad \ell \leq k-1, f \in B^+(E), t \geq 0, \tag{2}$$

it holds that

$$T_t^{(k)}[f](x) = T_t[f^k](x) + \int_0^t T_s\left[\beta \eta_{t-s}^{(k-1)}[f]\right](x)\,ds, \qquad t \geq 0, \tag{3}$$

where

$$\eta_{t-s}^{(k-1)}[f](x) = \mathcal{E}_x\left[\sum_{[k_1,\ldots,k_N]_k^2} \binom{k}{k_1,\ldots,k_N} \prod_{j=1}^N T_{t-s}^{(k_j)}[f](x_j)\right],$$

and $[k_1,\ldots,k_N]_k^2$ is the set of all non-negative N-tuples $(k_1,\ldots,k_N)$ such that $\sum_{i=1}^N k_i = k$ and at least two of the k_i are strictly positive.

INDUCTION: $k \mapsto k+1$

- Suppose the result is true for the first k moments.
- Recall $T_t[f](x) \to \langle f, \tilde{\varphi} \rangle \varphi(x)$ so that, for $k \geq 2$,

$$\lim_{t \to \infty} t^{-k} T_t[f^{k+1}](x) \to 0$$

- Hence:

$$\lim_{t \to \infty} t^{-k} T_t^{(k+1)}[f](x)$$

$$= \lim_{t \to \infty} t^{-k} \int_0^t T_s \left[\mathcal{E}. \left[\sum_{[k_1,\ldots,k_N]_{k+1}^2} \binom{k+1}{k_1,\ldots,k_N} \prod_{j=1}^N T_{t-s}^{(k_j)}[f](x_j) \right] \right] (x) \mathrm{d}s$$

$$= \lim_{t \to \infty} t^{-(k-1)} \int_0^1 T_{ut} \left[\mathcal{E}. \left[\sum_{[k_1,\ldots,k_N]_{k+1}^2} \binom{k+1}{k_1,\ldots,k_N} \prod_{j=1}^N T_{t(1-u)}^{(k_j)}[f](x_j) \right] \right] (x) \mathrm{d}u$$

$$= \lim_{t \to \infty} \int_0^1 T_{ut} \left[\mathcal{E}. \left[\sum_{[k_1,\ldots,k_N]_{k+1}^2} \binom{k+1}{k_1,\ldots,k_N} \frac{(t(1-u))^{k+1-\#\{j:k_j>0\}}}{t^{k-1}} \prod_{j=1}^N \frac{T_{t(1-u)}^{(k_j)}[f](x_j)}{(t(1-u))^{k_j-1}} \right] \right] (x) \mathrm{d}u$$

ROUGH OUTLINE OF THE INDUCTION: $k \mapsto k+1$

- From the last slide:

$$\lim_{t \to \infty} t^{-k} \mathrm{T}_t^{(k+1)}[f](x)$$

$$= \lim_{t \to \infty} \int_0^1 \mathrm{T}_{ut}\left[\mathcal{E} \cdot \left[\sum_{[k_1,\ldots,k_N]_{k+1}^2} \binom{k+1}{k_1,\ldots,k_N} \frac{(t(1-u))^{k+1-\#\{j:k_j>0\}}}{t^{k-1}} \prod_{j=1}^N \frac{\mathrm{T}_{t(1-u)}^{(k_j)}[f](x_j)}{(t(1-u))^{k_j-1}} \right] \right](x)\mathrm{d}u$$

- Largest terms in blue correspond to those summands for which $\#\{j : k_j > 0\} = 2$
- The induction hypothesis plus $\sum_{i=1}^N k_j = k+1$ ensures that the product term is asymptotically a constant
- The simple identity

$$\sum_{[k_1,\ldots,k_N]_{k+1}^2} \binom{k+1}{k_1,\ldots,k_N} \leq N^{k+1}$$

  shows us where the need for the hypothesis (H2) comes in.
- We need an ergodic limit theorem that reads (roughly): If

$$F(x,u) := \lim_{t \to \infty} F(x,u,t), \qquad x \in E, u \in [0,1],$$

  "uniformly" for $(u,x) \in [0,1] \times E$, then

$$\lim_{t \to \infty} \int_0^1 \mathrm{T}_{ut}[F(\cdot,u,t)](x)\mathrm{d}u = \int_0^1 \langle \tilde{\varphi}, F(\cdot,u) \rangle \mathrm{d}u$$

  "uniformly" for $x \in E$.

YAGLOM LIMITS

Theorem

Suppose that

- *(H1) holds (mean-semigroup ergodicity),*
- *the number of offspring is uniformly bounded by a constant N_{max},*
- *for all t sufficiently large*

$$\sup_{x \in E} \mathbb{P}_{\delta_x}(t < \zeta) < 1,$$

- *there exists a constant $C > 0$ such that for all $g \in B^+(E)$,*

$$\langle \tilde{\varphi}, \beta \mathbb{V}[g] \rangle \geq C \langle \tilde{\varphi}, g \rangle^2, \qquad where \qquad \mathbb{V}[g](x) = \beta(x)\mathcal{E}_x\left[\langle g, \mathcal{Z}\rangle^2 - \langle g^2, \mathcal{Z}\rangle\right]$$

Then

$$\lim_{t \to \infty} t\mathbb{P}_{\delta_x}(\zeta > t) = \frac{2\varphi(x)}{\langle \tilde{\varphi}, \beta \mathbb{V}[g]\rangle},$$

$$\lim_{t \to \infty} \mathbb{E}_{\delta_x}\left[\left.\left(\frac{\langle f, X_t\rangle}{t}\right)^k \right| \zeta > t\right] = k!\,\langle f, \tilde{\varphi}\rangle^k \left(\frac{\langle \tilde{\varphi}, \beta \mathbb{V}[g]\rangle}{2}\right)^k$$

and hence

$$\text{Law}\left(\left.\frac{\langle f, X_t\rangle}{t}\right| \zeta > t\right) \to \exp\left(\frac{2}{\langle \tilde{\varphi}, \beta \mathbb{V}[g]\rangle \langle f, \tilde{\varphi}\rangle}\right)$$

THEOREM: SUPERCRITICAL ($\lambda > 0$)

Suppose that (H1) holds along with (H2)$_k$ for some $k \geq 2$ and $\lambda > 0$. Redefine

$$\Delta_t^{(\ell)} = \sup_{x \in E, ||f|| \leq 1} \left| \varphi(x)^{-1} e^{-\ell \lambda t} T_t^{(\ell)}[f](x) - \ell! \langle f, \tilde{\varphi} \rangle^\ell L_\ell(x) \right|,$$

where $L_1(x) = 1$ and we define iteratively for $k \geq 2$,

$$L_k(x) = \int_0^\infty e^{-\lambda_* k s} \varphi(x)^{-1} \psi_s \left[\gamma \mathcal{E} . \left[\sum_{[k_1, \ldots, k_N]_k^2 \atop j:k_j > 0} \prod_{j=1}^N \varphi(x_j) L_{k_j}(x_j) \right] \right](x) \mathrm{d}s,$$

Then, for all $\ell \leq k$

$$\sup_{t \geq 0} \Delta_t^{(\ell)} < \infty \text{ and } \lim_{t \to \infty} \Delta_t^{(\ell)} = 0.$$

"At subcriticality the k-th moment scales like $e^{\lambda k t}$ (i.e. the first moment to the power k)"

THEOREM: SUBCRITICAL ($\lambda < 0$)

Suppose that (H1) holds along with (H2) for some $k \geq 2$ and $\lambda < 0$. Redefine

$$\Delta_t^{(\ell)} = \sup_{x \in E, ||f|| \leq 1} \left| \varphi(x)^{-1} e^{-\lambda t} T_t^{(\ell)}[f](x) - L_\ell \right|,$$

where we define iteratively $L_1 = \langle f, \tilde{\varphi} \rangle$ and for $k \geq 2$,

$$L_k = \tilde{\varphi}[f^k] + \int_0^\infty e^{-\lambda_* s} \tilde{\varphi} \left[\gamma \mathcal{E} \cdot \left[\sum_{[k_1, \ldots, k_N]_k^2} \binom{k}{k_1, \ldots, k_N} \prod_{\substack{j=1 \\ j:k_j > 0}}^N \psi_s^{(k_j)}[f](x_j) \right] \right] ds.$$

Then, for all $\ell \leq k$

$$\sup_{t \geq 0} \Delta_t^{(\ell)} < \infty \text{ and } \lim_{t \to \infty} \Delta_t^{(\ell)} = 0.$$

"At subcriticality the k-th moment scales like $e^{\lambda t}$ (i.e. like the first moment)"

WHAT ABOUT THE OCCUPATION MEASURE?

- Let us define the running occupation of the branching particle system via

$$\int_0^t X_s(\cdot)\mathrm{d}s, \qquad t \geq 0.$$

- What can we say about its moments?

$$\mathrm{M}_t^{(k)}[g](x) := \mathbb{E}_{\delta_x}\left[\left(\int_0^t \langle g, X_s\rangle \mathrm{d}s\right)^k\right], \qquad x \in E, g \in B^+(E), k \geq 1, t \geq 0.$$

- We know that the pair

$$\left(X_t, \int_0^t X_s\mathrm{d}s\right)$$

is Markovian and that its semigroup

$$\mathrm{v}_t[f,g] = \mathbb{E}_{\delta_x}\left[\mathrm{e}^{-\langle f, X_t\rangle - \int_0^t \langle g, X_s\rangle \, \mathrm{d}s}\right], \qquad t \geq 0,\, x \in E, f, g \in B^+(E),$$

solves

$$\mathrm{v}_t[f,g](x) = \hat{\mathrm{P}}_t[\mathrm{e}^{-f}](x) + \int_0^t \mathrm{P}_s\left[\mathrm{G}[\mathrm{v}_{t-s}[f,g]] - g\mathrm{v}_{t-s}[f,g]\right](x)\mathrm{d}s.$$

PLAYING THE SAME GAME AS BEFORE

Define a variant of the non-linear evolution equation associated with $(X_t, \int_0^s X_s \mathrm{d}s)$ via

$$\mathrm{u}_t[f,g](x) = \mathbb{E}_{\delta_x} \left[1 - \mathrm{e}^{-\langle f, X_t \rangle - \int_0^t \langle g, X_s \rangle \, \mathrm{d}s} \right], \qquad t \geq 0, \, x \in E, \, ||f|| < \infty, ||g|| < \infty.$$

For $f \in B_1^+(E)$, define

$$\mathrm{A}[f](x) = \beta(x)\mathcal{E}_x \left[\prod_{i=1}^{N}(1 - f(x_i)) - 1 + \sum_{i=1}^{N} f(x_i) \right], \qquad x \in E.$$

A re-arrangement of the joint semigroup of $(X_t, \int_0^t X_s \mathrm{d}s)$ is captured by:

Lemma
For all $f, g \in B^+(E)$, $x \in E$ and $t \geq 0$, the non-linear semigroup $\mathrm{u}_t[f,g](x)$ satisfies

$$\mathrm{u}_t[f,g](x) = \mathrm{T}_t[1 - \mathrm{e}^{-f}](x) - \int_0^t \mathrm{T}_s \left[\mathrm{A}[\mathrm{u}_{t-s}[f,g]] - g(1 - \mathrm{u}_{t-s}[f,g]) \right](x)\mathrm{d}s.$$

THEOREM: CRITICAL CASE ($\lambda = 0$)

Suppose that (H1) holds along with (H2) for $k \geq 2$ and $\lambda = 0$. Define

$$\Delta_t^{(\ell)} = \sup_{x \in E, ||g|| \leq 1} \left| t^{-(2\ell-1)} \varphi(x)^{-1} \mathrm{M}_t^{(\ell)}[g](x) - 2^{-(\ell-1)} \ell! \, \langle g, \tilde{\varphi} \rangle^\ell \langle \mathbb{V}[\varphi], \tilde{\varphi} \rangle^{\ell-1} L_\ell \right|,$$

where $L_1 = 1$ and L_k is defined through the recursion $L_k = (\sum_{i=1}^{k-1} L_i L_{k-i})/(2k-1)$.
Then, for all $\ell \leq k$

$$\sup_{t \geq 0} \Delta_t^{(\ell)} < \infty \text{ and } \lim_{t \to \infty} \Delta_t^{(\ell)} = 0.$$

THEOREM: SUPERCRITICAL CASE ($\lambda > 0$)

Suppose that (H1) holds along with (H2) for some $k \geq 2$ and $\lambda > 0$. Redefine

$$\Delta_t^{(\ell)} = \sup_{x \in E, ||g|| \leq 1} \left| \varphi(x)^{-1} \mathrm{e}^{-\ell \lambda t} \mathrm{M}_t^{(\ell)}[g](x) - \ell! \langle g, \tilde{\varphi} \rangle^{\ell} L_\ell(x) \right|,$$

where $L_1 = 1/\lambda$ and for $k \geq 2$ we define iteratively,

$$L_k(x) = \int_0^\infty \mathrm{e}^{-\lambda * ks} \varphi(x)^{-1} \psi_s \left[\gamma \mathcal{E}. \left[\sum_{[k_1, \ldots, k_N]_k^2} \prod_{\substack{j=1 \\ j:k_j > 0}}^{N} \varphi(x_j) L_{k_j}(x_j) \right] \right](x) \mathrm{d}s,$$

Then, for all $\ell \leq k$

$$\sup_{t \geq 0} \Delta_t^{(\ell)} < \infty \text{ and } \lim_{t \to \infty} \Delta_t^{(\ell)} = 0.$$

THEOREM: SUBCRITICAL CASE ($\lambda < 0$)

Suppose that (H1) holds along with (H2) for some $k \geq 2$ and $\lambda < 0$. Redefine

$$\Delta_t^{(\ell)} = \sup_{x \in E, ||g|| \leq 1} \left| \varphi(x)^{-1} \mathrm{M}_t^{(\ell)}[g](x) - \ell! \langle g, \tilde{\varphi} \rangle^\ell L_\ell(x) \right|,$$

where $||g|| < \infty$, $L_1 = 1/|\lambda|$ and for $k \geq 2$, the constants L_k are defined recursively via

$$L_k(x) = \int_0^\infty \varphi(x)^{-1} \psi_s \left[\gamma \mathcal{E}. \left[\sum_{[k_1, \ldots, k_N]_k^2} \binom{k}{k_1, \ldots, k_N} \prod_{\substack{j=1 \\ j:k_j > 0}}^N \varphi(x_j) L_{k_j}(x_j) \right] \right](x) \, \mathrm{d}s$$

$$- k \int_0^\infty \varphi(x)^{-1} \psi_s \left[g \varphi L_{k-1} \right](x) \, \mathrm{d}s.$$

Then, for all $\ell \leq k$

$$\sup_{t \geq 0} \Delta_t^{(\ell)} < \infty \text{ and } \lim_{t \to \infty} \Delta_t^{(\ell)} = 0.$$

WHAT ABOUT SUPERPROCESSES?

- A Markov process $X := (X_t : t \geq 0)$ on $M(E)$, the space of finite measures on Lusin space E, with $\mathbb{P} := (\mathbb{P}_\mu, \mu \in M(E))$.
- Transition semigroup

$$\mathbb{E}_\mu \left[e^{-\langle f, X_t \rangle} \right] = e^{-\langle V_t[f], \mu \rangle}, \qquad \mu \in M(E), f \in B^+(E),$$

where

$$V_t[f](x) = P_t[f](x) - \int_0^t P_s \left\langle \psi(\cdot, V_{t-s}[f](\cdot)) + \phi(\cdot, V_{t-s}[f]) \right\rangle | (x) \mathrm{d}s.$$

- Here ψ denotes the local branching mechanism

$$\psi(x, \lambda) = -b(x)\lambda + c(x)\lambda^2 + \int_{(0,\infty)} \left(e^{-\lambda y} - 1 + \lambda y \right) \nu(x, \mathrm{d}y), \quad \lambda \geq 0, \quad (4)$$

where $b \in B(E)$, $c \in B^+(E)$ and $(x \wedge x^2)\nu(x, \mathrm{d}y)$ is a bounded kernel from E to $(0, \infty)$, and ϕ is the non-local branching mechanism

$$\phi(x, f) = \beta(x)f(x) - \beta(x)\gamma(x, f) - \beta(x) \int_{M(E)^\circ} (1 - e^{-\langle f, \nu \rangle}) \Gamma(x, \mathrm{d}\nu),$$

where $\beta \in B^+(E)$, $\gamma(x, f)$ is a bounded function on $E \times B^+(E)$ and $\nu(1)\Gamma(x, \mathrm{d}\nu)$ is a bounded kernel from E to $M(E)^\circ := M(E) \setminus \{ 0 \}$ with

$$\gamma(x, f) + \int_{M(E)^\circ} \langle 1, \nu \rangle \Gamma(x, \mathrm{d}\nu) \leq 1.$$

WHAT ABOUT SUPERPROCESSES?

- Keep the same notation e.g.

$$\mathrm{T}_t^{(k)}[f](x) := \mathbb{E}_{\delta_x}[\langle f, X_t \rangle^k], \qquad x \in E, f \in B^+(E), k \geq 1, t \geq 0.$$

- Under the same first ergodic moment assumption (H1) and (H2)$_k$ replaced by

$$\sup_{x \in E} \left(\int_0^\infty |y|^k \nu(x, \mathrm{d}y) + \int_{M(E)^\circ} \langle 1, \nu \rangle^k \Gamma(x, \mathrm{d}\nu) \right) < \infty.$$

- A different proof is needed because we cannot work under the expectation with individual particles.

- Instead an approach using Faa di Bruno's formula can be used taking advantage of the smoother branching mechanism than in the particle setting.

- The same conclusions hold for the critical, supercritical and subcritical setting as for the branching particle setting, albeit the constants in the limit are slightly different.

Thank you!