Scaling the Aldous-Broder chain on the high-dimensional torus

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"Branching processes and their applications"
Angers, 26^{th} May 2023

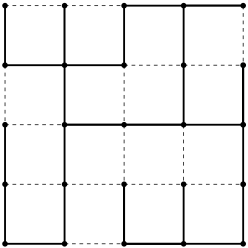
SPP2265
Random Geometric Systems

UNIVERSITÄT DUISBURG ESSEN
Offen im Denken

The uniform spanning tree

Given a finite connected graph $G = (V, E)$:

- A **spanning tree** is a connected subgraph $T = (V, E')$ with $E' \subseteq E$ without loops.



- The **UST** is uniformly distributed on the space of all spanning trees.

$\rightsquigarrow$ On the complete graph $\mathbb{K}_m$ with m vertices the UST equals in law the family tree of a *Bienaymé branching process with Poisson*(1) *offspring* law conditioned to have m vertices.

The Aldous-Broder algorithm

[Aldous (1990)], [Tsoucas (1989)], [Broder (1989)]

1. Consider the *RW* $(W(n))_{n\in\mathbb{N}_0}$ *on* G.
2. Let W run until it has seen all vertices, and put

$$E' := \{\{W(T_v - 1), v\};\ v \in V \setminus \{W(0)\}\},$$

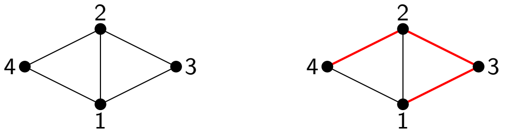
with

$$T_v := \inf\{n \geq 0 :\ W(n) = v\}$$

the **first** hitting time of v.

$\rightsquigarrow$ $G' = (V, E')$ is the uniform spanning tree.

Example. $W(0) = 3,\ W(1) = 2,\ W(2) = 3,\ W(3) = 1,\ W(4) = 2,\ W(5) = 4,\ \ldots$



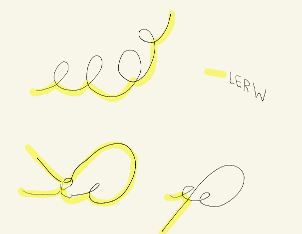
$\rightsquigarrow$ A computationally faster algorithm relies on Loop Erased Random Walks, [Wilson (1996)]

Loop-erased random walk (LERW)

[Lawler (1979)]

Let $\gamma([0, M]) := (\gamma(0), ..., \gamma(M))$, $M \in \mathbb{N}$, be a path on a locally finite connected graph $G = (V, E)$:

- Define the **loop erasure** $LE(\gamma([0, M]))$ by erasing loops in the order they appear.

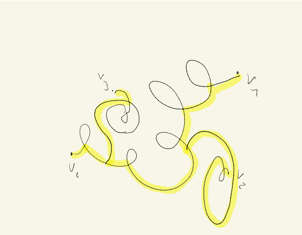


- A random walk $(W(n))_{n \in \mathbb{N}_0}$ on $G = (V, E)$ is the V-valued MC having jumps to every neighbor with the same probability.
- If W is transient, $LE(W([0, \infty)))$ well-defined even if $M = \infty$.

Wilson's algorithm

We can construct a UST on a finite connected graph $G = (V, E)$ as follows:

- Pick arbitrary vertices $v_0, v_1 \in V$, run a RW W starting in $W(0) = v_0$ until the first hitting time T_{v_1} and let $\mathcal{T}_1 := \mathrm{LE}\big(W([0, T_{v_1}])\big)$.
- Given $\mathcal{T}_k$, pick $v_{k+1} \in V$, run a RW W starting in $W(0) = v_{k+1}$ independently of what happened before until the first hitting time $T_{\mathcal{T}_k}$ and let $\mathcal{T}_{k+1} := \mathcal{T}_k \uplus \mathrm{LE}\big(W([0, T_{\mathcal{T}_k}])\big)$.



- Continue until all vertices are in the tree.

$\rightsquigarrow$ Distance between two randomly sampled leaves X_0 and X_1 in the UST equals in distribution the length of $\mathrm{LE}(W([0, T_{X_1}]))$ with $W(0) = X_0$.

[PERSONAL COMMUNICATION ALDOUS WITH DIACONIS]

Let RW W run from time $-\infty$ to time 0, and put

$$E' := \{\{v, W(L_v + 1)\};\ v \in V \setminus \{W(0)\}\},$$

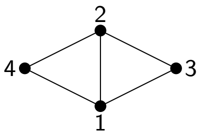
with

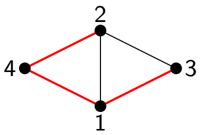
$$L_v := \max\{n < 0 :\ W(n) = v\}$$

the **last** hitting time of v.

$\rightsquigarrow$ $G' = (V, E')$ is the uniform spanning tree.

Example. ..., $W(-5) = 2$, $W(-4) = 3$, $W(-3) = 1$, $W(-2) = 2$, $W(-1) = 4$, $W(0) = 1$





The Aldous-Broder map

Let $\gamma = (\gamma(n))_{n\in\mathbb{N}}$ be a path on $G = (V, E)$:

- Define the **Aldous-Broder map** $\mathrm{AB}(\gamma)$ by letting for each $m \in \mathbb{N}$,
  - *vertex set.* $T(m) := \{\gamma(0), ..., \gamma(m)\}$,
  - *root.* $\varrho(m) := \gamma(m)$, and
  - *edge set.* $E(m) := \big\{\{v, \gamma(L_v + 1)\};\, v \in T(m-1)\big\}$.

  $\rightsquigarrow$ This yields a path with values in the space of rooted trees.

- If W is the RW on $G = (V, E)$, we refer to

$$X := \mathrm{AB}(W)$$

  as the **Aldous-Broder chain**.

$\rightsquigarrow$ Question for today. Scaling limit as graph size goes to ∞?

What is a tree?

Identify graph-theoretical trees $G = (T, E)$ with (finite) metric measure spaces $(T, \mathrm{d}_{gr}, \mu_{lf})$ where d_{gr} is the *graph distance* and μ_{lf} the *uniform distribution* on the set of leaves.

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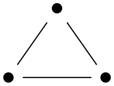
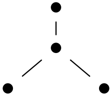


I am not a tree :-(



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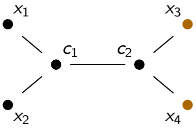

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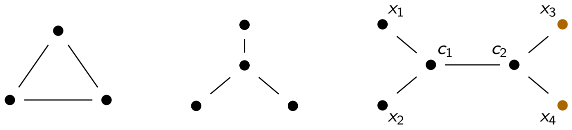

I am not a tree :-(I am a tree :-)

Definition

- A metric space (T, d) is called a **metric tree** if (T, d) is 0-hyperbolic and contains all branch points.
- A **metric measure tree** (T, d, μ) is a metric tree together with a probability measure μ.

$\rightsquigarrow$ $\mathbb{R}$-trees are path-connected metric trees.

Notions of convergence of metric (measure) spaces

- **Gromov-Hausdorff.** $(X_n, d_n)_{n\in\mathbb{N}} \xrightarrow[n\to\infty]{\text{GH}} (X, d)$ iff there are a metric space (Z, r) and isometries $\phi_n : X_n \to Z$, $n = 1, 2, ...,$ and $\phi : X \to Z$ with

$$d_H^{(Z,r)}\big(\phi_1(X_n), \phi(X)\big) \underset{n\to\infty}{\longrightarrow} 0.$$

- **Gromov-weak.** $(X_n, d_n, \mu_n)_{n\in\mathbb{N}} \xrightarrow[n\to\infty]{\text{Gw}} (X, d, \mu)$ iff there are a metric space (Z, r) and isometries $\phi_n : \mathrm{supp}(\mu_n) \to Z$, $n = 1, 2, ...,$ and $\phi : \mathrm{supp}(\mu) \to Z$ with

$$\big(\phi_n\big)_*\mu_n \underset{n\to\infty}{\Longrightarrow} \big(\phi\big)_*\mu.$$

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Useful facts and inclusions of notions of convergence

1. $(X_n, d_n, \mu_n)_{n\in\mathbb{N}} \xrightarrow[n\to\infty]{\text{GHw}} (X, d, \mu)$ implies

$$(X_n, d_n, \mu_n)_{n\in\mathbb{N}} \xrightarrow[n\to\infty]{\text{GH}} (X, d, \mu) \quad \text{and} \quad (X_n, d_n, \mu_n)_{n\in\mathbb{N}} \xrightarrow[n\to\infty]{\text{Gw}} (X, d, \mu).$$

2. [Greven, Pfaffelhuber & W. (2009)]
$(X_n, d_n, \mu_n)_{n\in\mathbb{N}} \xrightarrow[n\to\infty]{\text{Gw}} (X, d, \mu)$ iff

$$(d_n(U_i^n, U_j^n))_{1\le i<j\le k} \underset{n\to\infty}{\Longrightarrow} (d(U_i, U_j))_{1\le i<j\le k}$$

for all $k \in \mathbb{N}$, $U_1^n, \ldots, U_k^n$ i.i.d. $\sim \mu_n$ and $U_1, \ldots, U_k$ i.i.d. $\sim \mu$.

Scaling limit of the UST on the complete graph

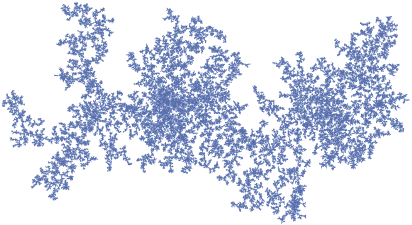
[Aldous (1993)]

Theorem (The CRT as scaling limit)

For a critical offspring law with variance $0 < \sigma^2 < \infty$, let T_N be the Bienaymé tree conditioned to have N nodes. Then

$$\frac{T_N}{\sqrt{\sigma^2 N}} \xrightarrow[N\to\infty]{GHw} \mathrm{CRT},$$

where CRT *is the* **Continuum Random Tree**.



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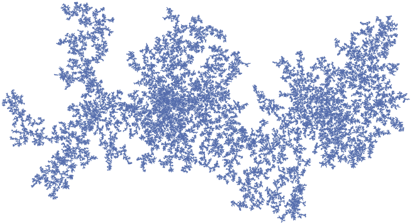
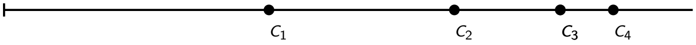


Corollary

The UST on the complete graph $\mathbb{K}_m$ with edge-lengths $1/\sqrt{m}$ converges GH-weakly in law to the CRT.

CRT as limit of growing trees

1. Let $(C_1, C_2, C_3, \ldots)$ be the times of a non-homogeneous Poisson point process with **rate** $r(t) = t$. In particular, $\mathbb{P}(C_1 > x) = e^{-\frac{x^2}{2}}$ (*Rayleigh distribution*).



$\rightsquigarrow$ The process that describes the *distance to the root* is the Rayleigh process.

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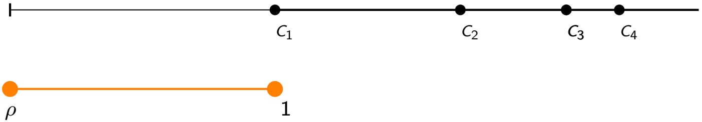

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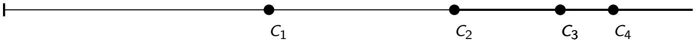
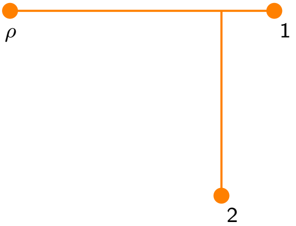
3. Inductively, obtain $\mathcal{R}(k+1)$ from $\mathcal{R}(k)$ by attaching an edge of length $C_{k+1} - C_k$ to a uniform random point of $\mathcal{R}(k)$ labeling a new leaf $k+1$.





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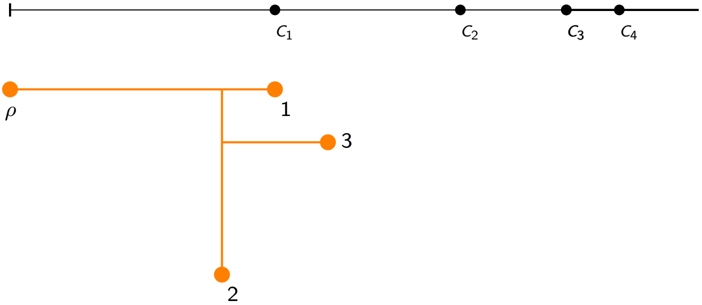

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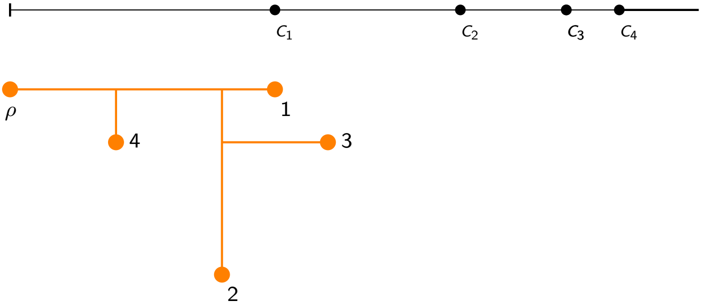


$\rightsquigarrow$ The process that describes the *distance to the root* is the Rayleigh process.

The CRT as scaling limit of the UST on tori

$$\mathbb{Z}_N^d := \{1, 2, ..., N\}^d\big|_{\mathrm{mod} N}$$

- [Benjamini & Kozma (2003)] $d \geq 5$

$$\mathbb{E}_{\pi_N}\left[\#\mathrm{LE}(W[0, T_0])\right] = \mathcal{O}\big(N^{\frac{d}{2}}\big)$$

- [Peres & Revelle (ArXiv2005)] $d \geq 5$. For all $x > 0$,

$$\mathbb{P}_{\pi_N}\big(\#\mathrm{LE}(W[0, T_0]) > x\beta(d)N^{\frac{d}{2}}\big) \underset{N\to\infty}{\longrightarrow} e^{-\frac{x^2}{2}}.$$

with $\beta(d) := \frac{\gamma(d)}{\sqrt{\alpha(d)}}$ where

$$\gamma(d) = \mathbb{P}\big(\mathrm{LE}(\hat{W}^1[0,\infty))) \cap \hat{W}^2([1,\infty)) = \emptyset\big)$$
$$\alpha(d) = \mathbb{P}\big(\mathrm{LE}(\hat{W}^1[0,\infty)) \cup \mathrm{LE}(\hat{W}^2[0,\infty)) \cap \hat{W}^3[1,\infty)) = \emptyset\big)$$

with $\hat{W}^1$, $\hat{W}^2$ and $\hat{W}^3$ independent RWs on $\mathbb{Z}^d$ starting in the origin.

Scaling limit of UST on the high dimensional torus

If $\mathbb{K}_m$ complete graph with m vertices, then $\frac{1}{\sqrt{m}}\text{UST}(K_m) \underset{m\to\infty}{\Longrightarrow} \text{CRT}$

$$\mathbb{Z}_N^d := \{1, 2, ..., N\}^d\big|_{\text{mod}N}$$

1. [Peres & Revelle (arXiv2005); $d \geq 5$] There is $\beta(d) \in (0, \infty)$ s.t.

$$\frac{1}{N^{\frac{d}{2}}}\text{UST}(\mathbb{Z}_N^d) \underset{N\to\infty}{\overset{\text{Gw}}{\Longrightarrow}} \beta(d) \cdot \text{CRT}.$$

2. [Schweinsberg (2009); $d = 4$] There exists a sequence (β_N) bounded away from zero and infinity such that

$$\frac{1}{\beta_N N^2 (\log N)^{\frac{1}{6}}}\text{UST}(\mathbb{Z}_N^4) \underset{N\to\infty}{\overset{\text{Gw}}{\Longrightarrow}} \text{CRT}.$$

3. [Archer, Nachmias & Shalev (arXiv2021); $d \geq 5$]

$$\frac{1}{N^{\frac{d}{2}}}\text{UST}(\mathbb{Z}_N^d) \underset{N\to\infty}{\overset{\text{GHw}}{\Longrightarrow}} \beta(d) \cdot \text{CRT}.$$

What is special about high dimensional tori?

- $(d \geq 3)$ [Spitzer (unpublished), Cox (1986)] Let $W = (W_n)_{n \in \mathbb{N}_0}$ be the RW on $\mathbb{Z}_N^d$, $d \geq 3$. Then

$$\mathcal{L}_{(1,0,\ldots,0)}\big(N^{-d} T_{(0,0,\ldots,0)}\big) \underset{N\to\infty}{\Longrightarrow} (1-\gamma(d))\delta_0 + \gamma(d)\mathcal{E}xp(1)$$

with $\gamma(d)$ **the escape probability** on $\mathbb{Z}_N^d$. Moreover,

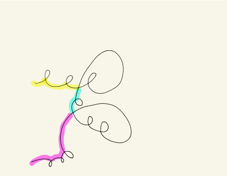
$$\mathcal{L}_{\pi_N}\big(N^{-d} T_{\underline{0}}\big) \underset{N\to\infty}{\Longrightarrow} \mathcal{E}xp\big((\gamma(d))^{-1}\big).$$

- $(d \geq 5)$ There is $\mathcal{O}(N^2 \log N) \ll (s_N) \ll \mathcal{O}(N^{\frac{d}{2}})$ with $\mathcal{L}_v(W(s_N)) = \pi_N$, $v \in \mathbb{N}$.
  - $\rightsquigarrow$ Separation (up to time $TN^{\frac{d}{2}}$, $T > 0$ fixed, w.o.p.) of short loops (lengths less than s_N) and long loops (length of order $\mathcal{O}(N^{\frac{d}{2}})$).

The scaling factor, $\beta(d)$, $d \geq 5$

There is a time scale $\mathcal{O}(N^2 \log N) \ll (s_N) \ll \mathcal{O}(N^{\frac{d}{2}})$ with $\mathcal{L}_v(W(s_N)) = \pi_N$, $v \in \mathbb{N}$.

- $\mathrm{LE}(W[0, T_v])$ decomposes into path segments segregated by the times at which a long loop occurs.



- Segments become asymptotically i.i.d. once you cut off pieces of length s_N in the beginning and in the end.
  - $\rightsquigarrow$ Want to know the number of segments and their length.
- **Concentration of length** If $s_N \ll r_N \ll N^{\frac{d}{2}}$ then $\mathbb{E}_{\pi_N}[\#\mathrm{LE}([0, r_N])] \sim \gamma_N(d) r_N$ with

$$\gamma_N(d) \underset{N\to\infty}{\longrightarrow} \mathbb{P}(\mathrm{LE}(\hat{W}^1([0,\infty))) \cap \hat{W}^2([1,\infty)) = \emptyset)$$

where $\hat{W}^1$, $\hat{W}^2$ independent RWs on $\mathbb{Z}^d$ starting in the origin.

A dynamic view on LERW

$\rightsquigarrow$ Scaling limit of Aldous-Broder chain as graph size goes to ∞?

$\mathbb{K}_m$ complete graph of size m

Let $W = (W(n))_{n\in\mathbb{N}_0}$ the symmetric RW on $\mathbb{K}_m$, and

$$Z(k) := \#\mathrm{LE}\big(W([0,k])\big), \qquad k \in \mathbb{N}_0.$$

Then $Z = (Z(k))_{k\in\mathbb{N}_0}$ is the $\mathbb{N}$-valued Markov chain with

$$\mathbb{P}\big(Z(k+1) = b | Z(k) = a\big) = \begin{cases} \frac{1}{m-1}, & \text{if } b \in \{1, ..., a-1\}, \\ \frac{m-1-a}{m-1}, & \text{if } b = a+1, \\ 0, & \text{else.} \end{cases}$$

$\rightsquigarrow$ Does this dynamics has a *scaling limit?*

Convergence to the Rayleigh process

Definition (Rayleigh process)

$R = (R(t))_{t\geq 0}$ is a $[0,\infty)$-valued piecewise deterministic Markov jump process such that

- with rate $R(t-)$ at time t, the Rayleigh process *jumps*

$$R(t-) \mapsto U \cdot R(t-),$$

  with U uniformly distributed on $[0,1]$,

- and in between jumps, R *increases linearly at unit speed*.

$$\Big(\tfrac{1}{\sqrt{m}}\#\mathrm{LE}^{\mathbb{K}_m}\big(W([0,\sqrt{m}t\rfloor])\big)\Big)_{t\geq 0} \underset{m\to\infty}{\Longrightarrow} \big(R(t)\big)_{t\geq 0},$$

weakly in Skorokhod space.

$\rightsquigarrow$ Rayleigh distribution is invariant; connection to CRT

Candidate for a scaling limit: Root growth with regrafting

[Evans, Pitman & Winter (2006)]

Definition (RGRG)

RGRG, is a piecewise deterministic jump process $R = (R(t))_{t \geq 0}$ s.t. given its current state (T, d, ϱ):

Regrafting. For each $v \in T$, a cut point occurs at unit rate $\mathrm{d}v$. As a result $T \setminus \{v\}$ is decomposed into two subtree components; one of them containing the root. We reconnect by identifying v with ϱ.

Root growth. In between two jumps the root grows away from the tree at unit speed.

$\rightsquigarrow$ The height of a tagged point follows a *Rayleigh process*.

Root growth with regrafting: Construction

[Evans, Pitman & Winter (2006)]

Remarks.

1. RGRG can be constructed using the points of a Poisson PP Π on

$$\Delta_+^2 := \{(x,y) \in \mathbb{R}_+^2 : x \in \mathbb{R}_+, y \in (0,x]\} \subseteq \mathbb{R}_+^2$$

   with intensity measure $\mathrm{d}x\mathrm{d}y$.

   For that we order the points of Π w.r.t. increasing first coordinates

$$(\tau_1(\Pi), p_1(\Pi)), (\tau_2(\pi), p_2(\Pi)), (\tau_3(\Pi), p_3(\Pi)), \ldots$$

   and use

   - $\tau_n(\Pi)$ for the time of the n^{th} regraft event, and
   - $p_n(\Pi)$ for the cut point associated with the n^{th} regraft event.

2. $\mathrm{RGRG}(t) \underset{t\to\infty}{\Longrightarrow} \mathsf{CRT}$.

3. The RGRG is also well-defined if started in an arbitrary rooted tree (possibly of infinite total length).

4. In particular, the RGRG has the CRT as its invariant distribution.

Scaling limit of the AB-chain on the complete graph

[Evans, Pitman & W. (2006)]

Theorem

If $(X^m)_{m\in\mathbb{N}}$ is a sequence of AB chains on the complete graph $\mathbb{K}_m$ and such that $(\frac{1}{\sqrt{m}}X^m(0))_{t\geq 0} \underset{m\to\infty}{\Longrightarrow} \mathrm{CRT}$, then

$$\Big(\frac{1}{\sqrt{m}}X^m(\sqrt{m}t)\Big)_{t\geq 0} \underset{N\to\infty}{\overset{GH}{\Longrightarrow}} \Big(RGRG(t)\Big)_{t\geq 0},$$

where the RGRG starts in the CRT.

A dynamic view in LERW: The high dimensional torus

$$\gamma(d) = \mathbb{P}\big(\mathrm{LE}(\hat{W}^1([0,\infty))) \cap \hat{W}^2([1,\infty)) = \emptyset\big)$$
$$\alpha(d) = \mathbb{P}\big((\mathrm{LE}(\hat{W}^1([0,\infty))) \cup \mathrm{LE}(\hat{W}^2([0,\infty))) \cap \hat{W}^3([1,\infty)) = \emptyset\big)$$

[Schweinsberg (2008)]

Theorem

If $R = (R(t))_{t\geq 0}$ is the Rayleigh process with $R(0) = 0$ and W is the RW on $\mathbb{Z}_N^d$, $d \geq 5$. Let for all $t \geq 0$,

$$Z^N(t) := \frac{1}{\beta(d) N^{\frac{d}{2}}} \#\mathrm{LE}\big(W([0, \lfloor (\alpha(d))^{-\frac{1}{2}} N^{\frac{d}{2}} t \rfloor])\big).$$

Then $Z^N = (Z^N(t))_{t\geq 0}$ converges to $R = (R(t))_{t\geq 0}$ in Skorokhod topology.

$\rightsquigarrow$ Can this be extended to a tree-valued dynamics?

Scaling the AB-chain on the torus: Conjecture

Conjecture (Angtuncio-Hernández, Berzunza-Ojeda & W.)

If $(X^N)_{N\in\mathbb{N}}$ is a sequence of AB chains on $\mathbb{Z}_N^d$, $d \geq 5$, starting in the $UST(\mathbb{Z}^d)$ then

$$\left(\frac{1}{N^{\frac{d}{2}}} X^N(N^{\frac{d}{2}}t)\right)_{t\geq 0} \underset{N\to\infty}{\overset{GH}{\Longrightarrow}} \left(\beta(d)\cdot RGRG((\alpha(d))^{\frac{1}{2}}t)\right)_{t\geq 0},$$

where $\mathrm{RGRG}(0) \overset{(d)}{=} \mathrm{CRT}$.

Scaling the AB-chain on the torus: main result

Trimming operator. $\Upsilon_\eta\big((T,d,\varrho)\big) := (T_\eta, d_\eta, \varrho)$, $\eta > 0$, defined as with

$$T_\eta := \{y \in T : \exists z \in T \text{ with } y \in [\varrho, z], d(y,z) \geq \eta\} \cup \{\varrho\}.$$

Our conjecture implies that for all $\eta > 0$,

$$\Big(\Upsilon_\eta\big(N^{-\frac{d}{2}} X^N(N^{\frac{d}{2}} t)\big)\Big)_{t\geq 0} \underset{N\to\infty}{\overset{\text{GH}}{\Longrightarrow}} \Big(\Upsilon_\eta\big(\beta(d)\cdot \mathrm{RGRG}((\alpha(d))^{\frac{1}{2}} t)\big)\Big)_{t\geq 0}.$$

$\rightsquigarrow$ If the conjecture holds, it also holds if $X^N(0) = (\{\varrho\}, \varrho)$ and RGRG(0) is the trivial rooted tree.

Theorem (Angtuncio-Hernández, Berzunza-Ojeda & W.)

If $(X^N)_{N\in\mathbb{N}}$ is a sequence of AB chains on $\mathbb{Z}_N^d$, $d \geq 5$, starting in the trivial tree then

$$\Big(N^{-\frac{d}{2}} X^N(\lfloor N^{\frac{d}{2}} t\rfloor)\Big)_{t\geq 0} \underset{N\to\infty}{\overset{GH}{\Longrightarrow}} \Big(\beta(d)\cdot RGRG((\alpha(d))^{\frac{1}{2}} t)\Big)_{t\geq 0},$$

where the RGRG also starts in the trivial rooted tree.

Proof Strategy

Theorem (Angtuncio-Hernández, Berzunza-Ojeda & W.)

If $(X^N)_{N\in\mathbb{N}}$ *is a sequence of AB chains on* $\mathbb{Z}_N^d$, $d \geq 5$, *starting in the trivial tree then*

$$\Big(N^{-\frac{d}{2}} X^N(\lfloor N^{\frac{d}{2}} t\rfloor)\Big)_{t\geq 0} \underset{N\to\infty}{\overset{GH}{\Longrightarrow}} \Big(\beta(d)\cdot RGRG((\alpha(d))^{\frac{1}{2}} t)\Big)_{t\geq 0},$$

where the RGRG also starts in the trivial rooted tree.

General strategy. Couple the RW driving the AB-chain and the Poisson PP driving the RGRG such that for each $T>0$ w.h.p.,

$$d_{\text{Skorohod}}\bigg(\Big(N^{-\frac{d}{2}} X^N(\lfloor N^{\frac{d}{2}} t\rfloor)\Big)_{t\geq 0}, \Big(\beta(d)\cdot \text{RGRG}((\alpha(d))^{\frac{1}{2}} t)\Big)_{t\in[0,T]}\bigg) \underset{N\to\infty}{\longrightarrow} 0.$$

Proof ideas: short loops don't change much

Choose a sequence $(s_N) \uparrow \infty$ such that $N^2(\log N)^2 \ll s_N \ll N^{\frac{d}{2}}$.

- Note that for all $t \geq 0$,

$$d_{\mathrm{H}}\Big(\Upsilon_{s_N N^{-\frac{d}{2}}}\big(N^{-\frac{d}{2}} X^N(\lfloor N^{\frac{d}{2}} t \rfloor)\big), N^{-\frac{d}{2}} X^N(\lfloor N^{\frac{d}{2}} t \rfloor)\Big) \leq s_N N^{-\frac{d}{2}} \underset{N\to\infty}{\longrightarrow} 0$$

  and that

$$\Upsilon_{s_N N^{-\frac{d}{2}}}\big(N^{-\frac{d}{2}} X^N(\lfloor N^{\frac{d}{2}} t \rfloor)\big) = N^{-\frac{d}{2}} \Upsilon_{s_N}\big(X^N(\lfloor N^{\frac{d}{2}} t \rfloor)\big).$$

  $\rightsquigarrow$ It is enough to show that

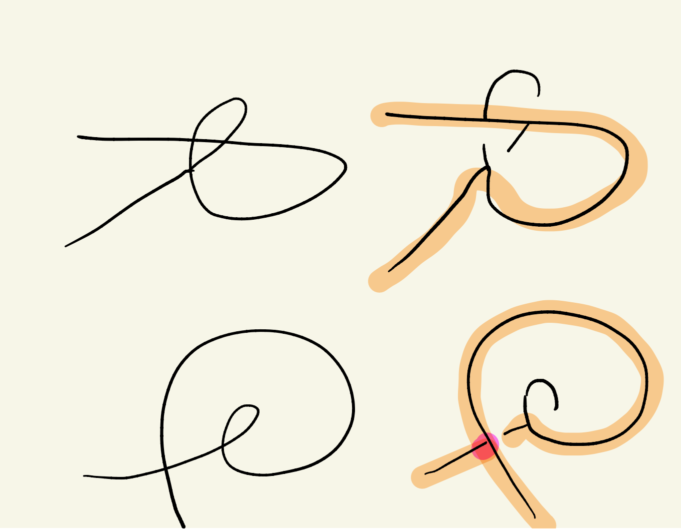
$$\Big(N^{-\frac{d}{2}} \Upsilon_{s_N}\big(N^{-\frac{d}{2}} X^N(\lfloor N^{\frac{d}{2}} t \rfloor)\big)\Big)_{t\geq 0} \underset{N\to\infty}{\overset{\text{GH}}{\Longrightarrow}} \Big(\beta(d)\cdot \text{RGRG}((\alpha(d))^{\frac{1}{2}} t)\Big)_{t\geq 0}$$

- Fix $T > 0$. With overwelming probability, for all $t \in [0, T]$,

$$\Upsilon_{s_N}\big(X^N(\lfloor N^{\frac{d}{2}} t \rfloor)\big) = \mathrm{LE}_{s_N}(W[0, t]),$$

  where $\mathrm{LE}_{s_N}(\gamma)$ denotes the local loop erasure of γ which only erases short loops of length $\leq s_N$.

Illustrating the role of short and long loops



Proof ideas: long loops are rare

Choose sequences $(s_N) \uparrow \infty$ and $(r_N) \uparrow \infty$ such that

$$\mathcal{O}\big(N^2 \log N\big) \ll s_N \ll r_N \ll N^{\frac{d}{2}}.$$

Let $\hat{W}^1$ and $\hat{W}^2$ be two independent RWs on $\mathbb{Z}_N^d$, and put

$$\alpha_N(d) := \mathbb{P}_{\pi_N \times \pi_N}\big(\mathrm{LE}\big(\hat{W}^1([0, r_N])\big) \cap \mathrm{Range}\big(\hat{W}^2([1, r_N])\big) \neq \emptyset\big).$$

Lemma (Peres & Revelle (2004ArXiv))

$$\alpha(d) := \lim_{N\to\infty} \frac{N^d}{(r_N)^2} \alpha_N(d)$$
$$= \mathbb{P}\big((\mathrm{LE}((\hat{W}^1[0, \infty)) \cup \mathrm{LE}(\hat{W}^2[0, \infty)) \cap \hat{W}^3[1, \infty) = \emptyset\big),$$

with $\hat{W}^1$, $\hat{W}^2$ and $\hat{W}^3$ independent RWs on $\mathbb{Z}^d$ starting in the origin.

Proof ideas: approximation and coupling

1. Decompose the RW-path $W[0, TN^{\frac{d}{2}}]$ into path segments of length $\sim r_N$:

$$A_j^{s_N, r_N} := W\big[\{(j-1)r_N + s_N, ..., jr_N - s_N\}\big]; \qquad j = 1, ..., \lfloor \frac{TN^{\frac{d}{2}}}{r_N} \rfloor,$$

and refer to (i, j) as an $(W, (s_N, r_N))$-**regraft index** if

$$\mathrm{LE}(A_j^{s_N, r_N}) \cap \mathrm{Range}(A_i^{s_N, r_N}) \neq \emptyset.$$

2. Discretize the Poisson PP as follows:

$$\big\{(\lfloor x(\alpha_N(d))^{-\frac{1}{2}} \rfloor \alpha_N(d)^{\frac{1}{2}}, \lfloor x(\alpha_N(d))^{-\frac{1}{2}} \rfloor \alpha_N(d)^{\frac{1}{2}}); \, (x, y) \in \Pi\big\},$$

and refer to (i, j) **as an** $(\Pi, \alpha_N(d))$-**regraft index** if

$$\Pi\big([i\alpha_N(d)^{\frac{1}{2}}, (i+1)\alpha_N(d)^{\frac{1}{2}}) \times [j\alpha_N(d)^{\frac{1}{2}}, (j+1)\alpha_N(d)^{\frac{1}{2}})\big) \geq 1.$$

3. Show that we can couple W and Π such that w.o.p. for all $j \leq i \leq \frac{TN^{\frac{d}{2}}}{r_N}$,$(i, j)$ as an $(W, (s_N, r_N))$-regraft index iff (i, j) as an $(\Pi, \alpha_N(d))$-regraft index.

$\rightsquigarrow$ Consequently, for all $T > 0$ w.o.p. up to time $TN^{\frac{d}{2}}$,

- branching events happen at the same time, and
- yield trees of the same tree shape,

while in between two regraft events we rely on [Schweinsberg (2008)].

La fin

Merci de votre attention