In and out of zero for a critical Bienaymé-Galton-Watson process with immigration

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Model

Let $(Z_m)_{m\geq 0}$ be a Bienaymé-Galton-Watson process with immigration with reproduction law μ and immigration law ν, both on $\mathbb{N}_0$. Assume μ has a generating function given (for $|s|\leq 1$) by

$$f(s) := s + c(1-s)^{1+\alpha}L(1-s), \tag{1}$$

and the immigration law ν has a generating function given (for $|s|\leq 1$) by

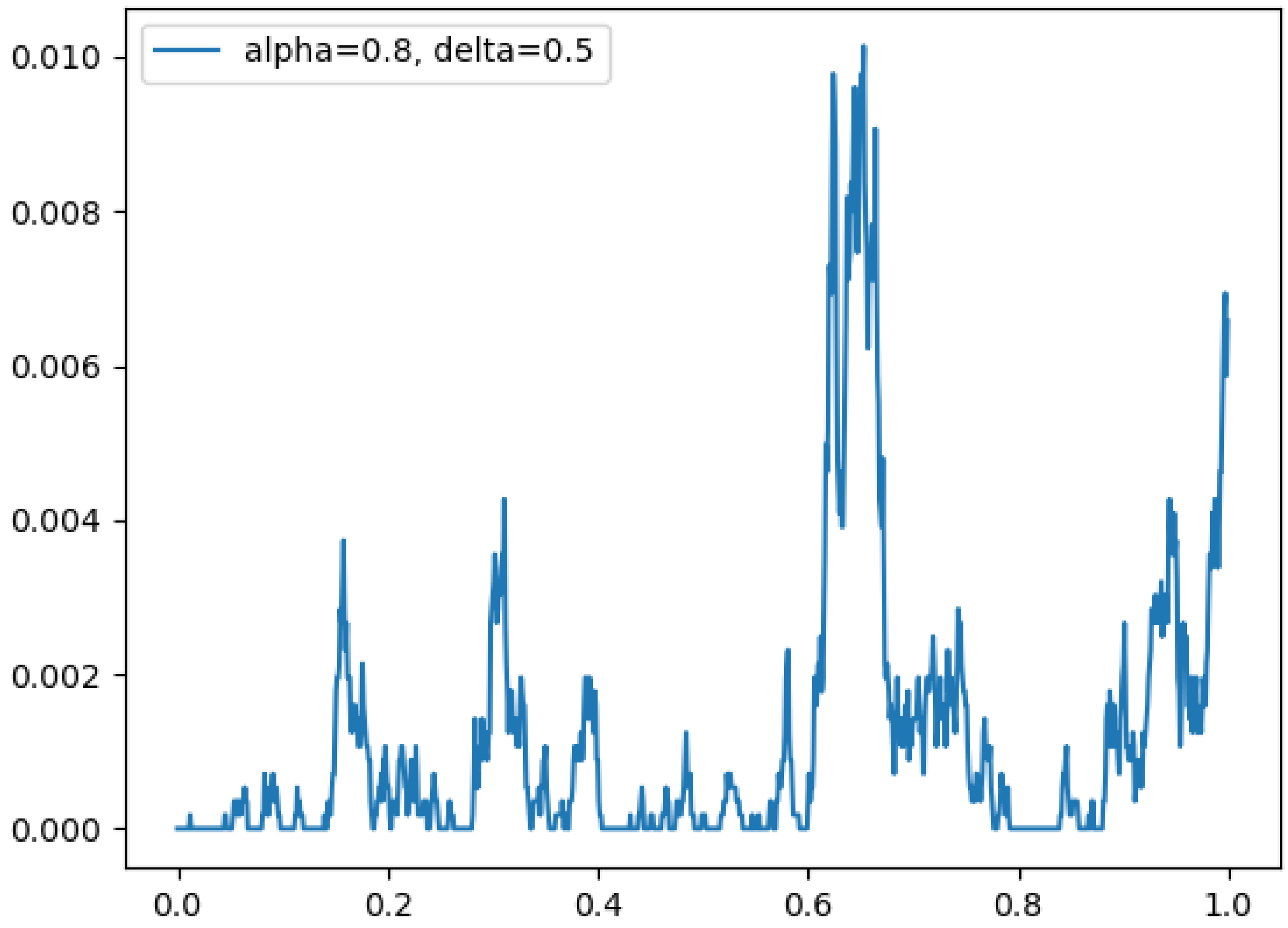
$$h(s) := 1 - d(1-s)^{\alpha}G(1-s), \tag{2}$$

where $c,\ d\geq 0$ and $\alpha\in(0,1]$ and $L(x)\sim G(x)$ for $x\to 0$ and L is slowly varying at 0.
Define a scaled process

$$X^{(n)}_t := \frac{1}{b_n}Z_{[nt]} \quad\text{with}\quad b_n^{\alpha}\sim nL(1/b_n) \text{ as } n\to\infty. \tag{3}$$

By Kawazu and Watanabe 1971 the limit of (3) exists and it is a CBI-process $(X_t)_{t\geq 0}$ with the Laplace transform given by

$$\mathbb{E}_{x_0}e^{-\lambda X_t} = (1+\alpha c\lambda^{\alpha}t)^{-\frac{d}{\alpha c}}\exp\left(\frac{-\lambda x_0}{(1+\alpha c\lambda^{\alpha}t)^{1/\alpha}}\right). \tag{4}$$



δ-assumption

Let

$$\delta := \frac{d}{\alpha c}.$$

In Foucart and Bravo 2014 the zero-level set of the scaling limit of X, i.e. of a CBI-process, was characterised, in particular,

- when $\delta < 1$ the level 0 is recurrent;
- when $\delta\geq 1$ it is polar.

Convergence of local time at 0

Let $(L)_{t\geq 0}$ be the local time of X at 0 and for $t\geq 0$ let the "naïve" local time of $X^{(n)}$ at 0 be

$$L^{(n)}_t := \#\left\{j\leq [nt] : X^{(n)}_{j/n} = 0\right\}.$$

Theorem 1. *Assume that $\delta\in(0,1)$. Then there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$, such that*

$$(X^{(n)}, L^{(n)}/a_n) \xrightarrow{w} (X, L) \text{ as } n\to\infty.$$

The sequence a_n satisfies

$$a_n^{1-\delta}\sim nL^*(a_n) \text{ as } n\to\infty,$$

for a slowly varying (at infinity) function L^*.

Simulation

Consider the case $\alpha < 1$.

Claim 1. *Let $\eta_{\alpha}^{d,G}$ be an α-stable random variable such that*

$$\mathbb{E}\exp(-\lambda\eta_{\alpha}^{d,G})\sim 1 - d\lambda^{\alpha}G(\lambda),\ \lambda\to 0+,$$

then the immigration

$$I \stackrel{d}{=} I^{d,G} := Pois(\eta_{\alpha}^{d,G}),$$

and upon setting $I' = I^{c(1+\alpha),L_1}$ the regeneration

$$R \stackrel{d}{=} Be(I')(I'+1),$$

where L_1 is such that $L_1(x)\sim L(x)$ when $x\to 0+$.

Yaglom limit

Let $(W_m)_{m\geq 0}$ be an excursion of Z above 0, that is assume that m^* is such that $Z(m^*-1) = 0$ and $Z(m^*) > 0$, then $(W_m)_{m\geq 0}$ is defined as

$$W(0) = Z(m^*) \quad\text{and}\quad W(m+1) = \begin{cases} Z(m^*+m+1), & \text{if } W(m) > 0,\\ 0, & \text{otherwise.}\end{cases}$$

The length of excursion W is defined as $\rho := \inf\{m > 0 : W(m) = 0\}$. Note that $\mathbb{P}(\rho > n) = \mathbb{P}(W(n) > 0)$.

Theorem 2. *Let L^* be a slowly varying (at infinity) function. For $\delta > 0$ it holds that*

$$\mathbb{P}(Z_n = 0)\sim n^{-\delta}L^*(n) \text{ as } n\to\infty. \tag{5}$$

Theorem 3. *Let L^* be a slowly varying (at infinity) function. For $\delta > 0$ it holds that $\mathbb{E}\rho = \infty$ and*
for $\delta\leq 1$

$$\mathbb{P}(\rho > n)\sim \frac{L^*(n)}{n^{1-\delta}} \text{ as } n\to\infty.$$

for $\delta > 1$

$$\lim_{n\to\infty}\mathbb{P}(\rho > n) = \mathbb{P}(\rho = \infty) > 0.$$

Theorem 4. *For $\lambda\geq 0$ and $\delta > 0$*

$$\lim_{n\to\infty}\mathbb{E}\left(\exp\left(-\lambda\frac{W(n)}{b_n}\right)\middle| W(n) > 0\right) = \frac{1}{(1+\alpha c\lambda^{\alpha})^{\delta\vee 1}}. \tag{6}$$

Reflection on the infinite variance

$$\left\{\text{Finite variance of reproduction}\right\} = \left\{\alpha = 1 \text{ and } L(0) < \infty\right\}.$$

From Zolotarev 1957-Slack 1968 it is readily seen that for a BGW without immigration which satisfies (1), denote it $\tilde{Z}$, it holds that

$$\lim_{n\to\infty}\mathbb{E}\left(\exp\left(-\lambda\frac{\tilde{Z}(n)}{b_n}\right)\middle|\tilde{Z}(n) > 0\right) = 1 - \left(\frac{\alpha c\lambda^{\alpha}}{1+\alpha c\lambda^{\alpha}}\right)^{1/\alpha}.$$

Upon assuming the finite variance the known results are recovered, see Yaglom 1947 and Vatutin 1977. Now both limit laws are exponential and their Laplace transforms are $\frac{1}{1+c\lambda}$.