Consistent least squares estimation in population-size-dependent branching processes

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joint work with

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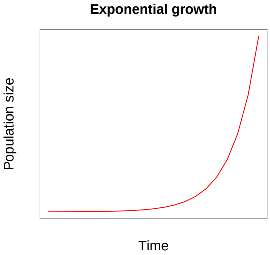
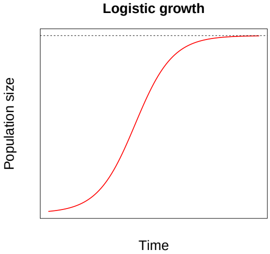
Biological motivation

Logistic growth

- The standard assumption of independence between individuals in a population leads to exponential growth.
- This assumption is often not realistic: an individual's lifetime and reproductive parameters usually depend on the availability of a number of resources, such as food, habitat, and breeding opportunities.
- A population can grow only until it reaches the maximum population size a particular habitat can support, named the carrying capacity.





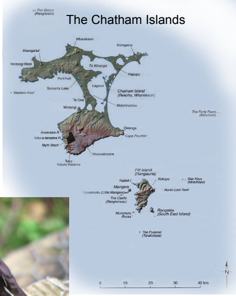
Reindeers at Saint Matthew Island

- In 1944, 29 reindeer were introduced to St. Matthew Island by the United States Coast Guard.
- The Coast Guard abandoned the island a few years later, leaving the reindeer.
- The reindeer population rose to about 6,000 by 1963.
- In the next two years, the number declined to 42 animals (41 females and one male).
- By the 1980s, the reindeer population had died out.

Fig: Source: `https://www.adn.com/features/article/what-wiped-out-st-matthew-islands-reindeer/2010/01/17/`

The black robin population

The endangered Chatham Island black robin population (New Zealand) was saved from the brink of extinction in the early 1980s.



The black robin population

- The current growth of the population does not appear to be exponential.
- The population has not yet reached the carrying capacity of the island.
- A low estimated value of the carrying capacity would highlight the need to find further appropriate habitat.

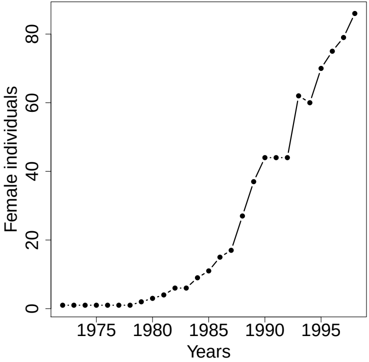


Fig: Number of adult females between 1972 and 1998.

Aim

To estimate the carrying capacity of the black robin population.

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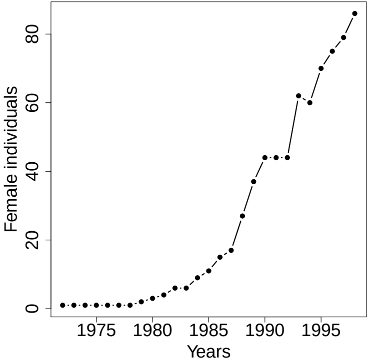

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Outline

Population-size-dependent branching processes (PSDBPs)

- $\xi(z)$: the offspring distribution at population size z, $\quad z \geq 1$
- Z_n : the population size at generation n,

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_{n,i}(Z_n), \qquad n \geq 0$$

where conditionally on Z_n, the random variables $\xi_{n,i}(Z_n)$ are i.i.d.

- $m(z) := \mathbb{E}[\xi(z)]$ the offspring mean at population size z
- $\sigma^2(z) := \mathrm{Var}[\xi(z)]$ the offspring variance at population size z
- The conditional mean and variance of the process are given by

$$\mathbb{E}[Z_n \mid Z_{n-1}] = Z_{n-1}\, m(Z_{n-1}), \quad \mathrm{Var}[Z_n \mid Z_{n-1}] = Z_{n-1}\, \sigma^2(Z_{n-1})$$

Mathematical carrying capacity

The carrying capacity K of a population in a certain environment is defined such that

$$m(z) > 1, \quad \text{if } z < K, \quad \text{and} \quad m(z) < 1, \quad \text{if } z > K.$$

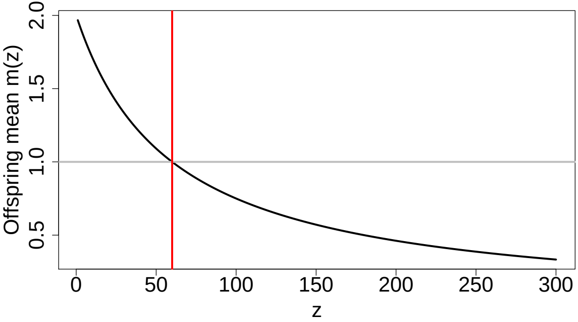

Fig: Mean offspring function $m(z)$ of a PSDBP with carrying capacity $K = 60$.

Almost sure extinction!

Carrying capacity + random nature of the process = a.s. extinction.

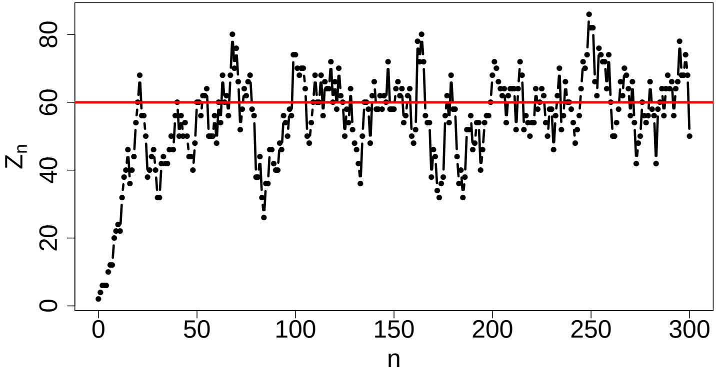

Fig: Portion of a trajectory of a PSDBP with carrying capacity $K = 60$.

A PSDBP model for the black robin population

Between generations n and $n+1$, if there are z female birds,

- A female bird makes a successful attempt to reproduce with probability $r := r(z, K, v)$, where

$$r(z,K,v) = \frac{v\,K}{(\mu-1)z+K} \qquad \text{(Beverton-Holt model).}$$

$$r(z,K,v) = v\left(\frac{1}{\mu}\right)^{z/K} \qquad \text{(Ricker model).}$$

  with $\mu = 5pv/d$.

- If reproduction is successful, the mother produces daughters according to a binomial distribution with $n = 5$ and $p = 0.1988$.

- The mother survives to the next generation with probability $1 - d = 0.6861$.

Outline

Parametric estimation in PSDBPs

Given that our goal is to estimate the carrying capacity, we will need a parametric PSDBP model.

We assume that the offspring distribution belongs to some parametric family, that is,

$$p_k(z) := \mathbb{P}[\xi(z) = k] = p_k(z, \boldsymbol{\theta}_0),$$

for some $\boldsymbol{\theta}_0 \in int(\Theta) \subseteq \mathbb{R}^b$. Then, $m(z) = m(z, \boldsymbol{\theta}_0)$.

In the black robin population example the parameter is $\boldsymbol{\theta}_0 = (K, v)$.

We aim at finding C-consistent estimators for θ_0 based on the observation of the population sizes $Z_0, Z_1, Z_2, \ldots, Z_n$.

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A first approach: MLE of the offspring mean

First aim: to find a *good* estimator for the offspring means $m(z)$ based on the observation of the population sizes $Z_0, Z_1, Z_2, \ldots, Z_n$ in a general framework.

- MLE for the offspring mean $m(z)$ at population size z, based on the observation of the population sizes $Z_0, Z_1, \ldots, Z_n$:

$$\hat{m}_n(z) := \frac{\sum_{i=1}^{n} Z_i \mathbb{1}_{\{Z_{i-1}=z\}}}{z \sum_{i=1}^{n} \mathbb{1}_{\{Z_{i-1}=z\}}}.$$

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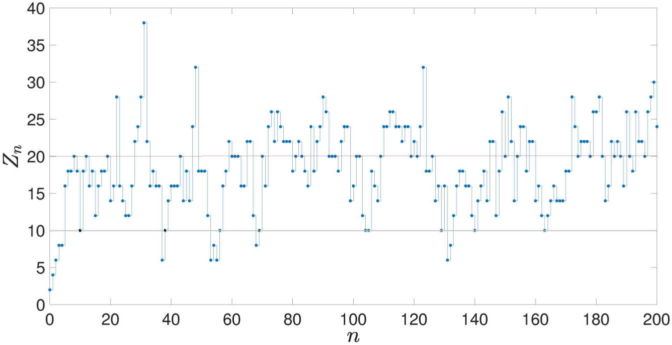

Fig: Example with carrying capacity $K = 20$

$$\hat{m}_n(z) := \frac{\sum_{i=1}^{n} Z_i \, \mathbb{1}_{\{Z_{i-1}=z\}}}{z \sum_{i=1}^{n} \mathbb{1}_{\{Z_{i-1}=z\}}}$$

Example:

$\hat{m}_{200}(10) = (18+14+16+20+10+18+16+14+12)/(10 \cdot 9) = 138/90 = 1.53$

MLE of the offspring mean: asymptotic properties

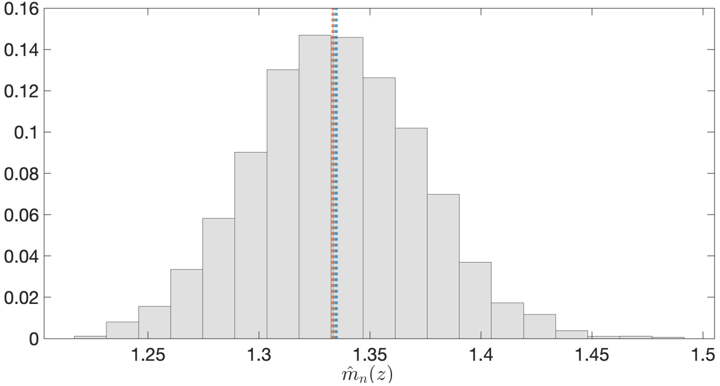

Fig: Histogram of $\hat{m}_n(z)$ for $z = 10$ and $n = 2000$, based on 5000 simulations, $K = 20$.

Real value of $m(z) = 1.3333$; Empirical mean of $\hat{m}_n(z) = 1.3349$

Conditional on $Z_n > 0$, $\hat{m}_n(z) \to m^{\uparrow}(z) = 1.3334 \neq m(z)$ as $n \to \infty$

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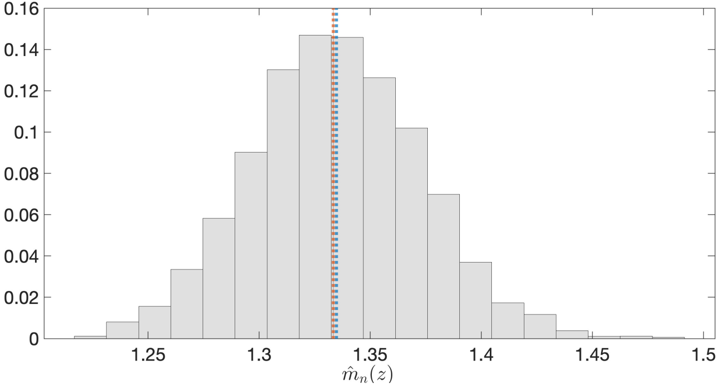

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How do $m(z)$ and $m^{\uparrow}(z)$ differ?

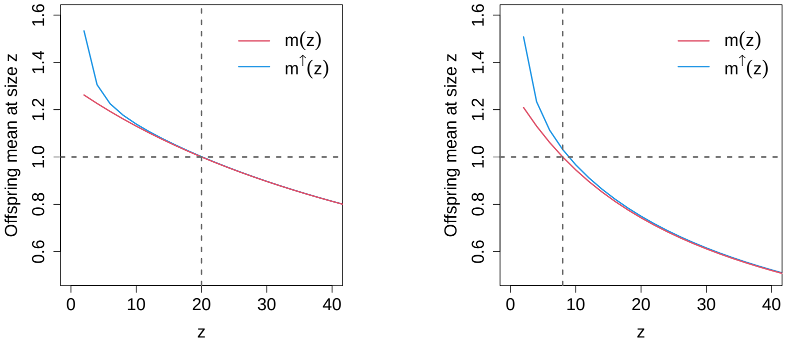

Fig: Comparison between the functions $z \mapsto m(z)$ (red line) and $z \mapsto m^{\uparrow}(z)$ (blue line). Left: $K = 20$. Right: $K = 8$.

Conditioning on $Z_n > 0$ — The Q-process

- Q : the sub-stochastic transition probability matrix of $\{Z_n\}$ restricted to the transient states $\{1, 2, \ldots\}$
- We set the following conditions:
  - (A1) The matrix Q is irreducible
  - (A2) $\limsup_{z\to\infty} m(z) < 1$
  - (A3) For each $\nu \in \mathbb{N}$, $\sup_{z\in\mathbb{N}} E[\xi_{01}(z)^{\nu}] < \infty$.
- Under these conditions,
  - $\mathbb{P}[Z_n \to 0] = 1$ (almost sure extinction)
  - $Q^n \sim \rho^n \boldsymbol{v}\boldsymbol{u}^\top$, where $\rho := \lim_{n\to\infty}(Q^n)_{ij}^{1/n}$, and $\boldsymbol{u}, \boldsymbol{v} > \mathbf{0}$ such that

$$\boldsymbol{u}^\top Q = \rho \boldsymbol{u}^\top, \quad Q\boldsymbol{v} = \rho\boldsymbol{v}, \quad \boldsymbol{u}^\top\mathbf{1} = 1, \quad \text{and} \quad \boldsymbol{u}^\top\boldsymbol{v} = 1.$$

Conditioning on $Z_n > 0$ — The Q-process

- For n fixed: the process $\{Z_\ell\}_{0\leq\ell\leq n}$ conditioned on $Z_n > 0$ is a time-inhomogeneous Markov chain:

$$\begin{aligned}\mathbb{P}[Z_\ell^{(n)} = j \mid Z_{\ell-1}^{(n)} = i] &:= \mathbb{P}[Z_\ell = j \mid Z_{\ell-1} = i,\ Z_n > 0] \\ &= Q_{ij}\,\frac{\boldsymbol{e}_j^\top Q^{n-\ell}\mathbf{1}}{\boldsymbol{e}_i^\top Q^{n-\ell+1}\mathbf{1}}.\end{aligned}$$

- As $n \to \infty$:

$$\begin{aligned}\mathbb{P}[Z_\ell^{\uparrow} = j \mid Z_{\ell-1}^{\uparrow} = i] &:= \lim_{n\to\infty} \mathbb{P}[Z_\ell = j \mid Z_{\ell-1} = i,\ Z_n > 0] \\ &= \lim_{n\to\infty} Q_{ij}\,\frac{\boldsymbol{e}_j^\top \rho^{n-\ell}\boldsymbol{v}}{\boldsymbol{e}_i^\top \rho^{n-\ell+1}\boldsymbol{v}} \\ &= Q_{ij}\,\frac{v_j}{\rho v_i}.\end{aligned}$$

$\{Z_\ell^{\uparrow}\}_{\ell\geq 0}$ is a positive recurrent time-homogeneous Markov chain called the Q-process

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The Q-process and 'Q-consistency' of the MLE

$m(z) := \mathbb{E}[\xi(z)]$ the mean offspring at population size z

$$\hat{m}_n(z) := \frac{\sum_{i=1}^{n} Z_i \, \mathbb{1}_{\{Z_{i-1}=z\}}}{z \sum_{i=1}^{n} \mathbb{1}_{\{Z_{i-1}=z\}}}$$

- In $\{Z_n\}$: $\quad m(z) = z^{-1} \sum_{j \geq 1} j \, Q_{zj}$
- In $\{Z_n^{\uparrow}\}$: $\quad m^{\uparrow}(z) = z^{-1} \sum_{j \geq 1} j \, Q_{zj}^{\uparrow} \qquad$ with $Q_{ij}^{\uparrow} := Q_{ij} \frac{v_j}{\rho v_i}$

Theorem (Braunsteins, Hautphenne, M., 2022a)

Under (A1)–(A3), for any $z \in \mathbb{N}$, initial state i, and $\varepsilon > 0$, $\hat{m}_n(z)$ satisfies

$$\lim_{n \to \infty} \mathbb{P}_i[|\hat{m}_n(z) - m^{\uparrow}(z)| > \varepsilon \,|\, Z_n > 0] = 0 \qquad \text{'Q-consistency'}$$

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Asymptotic normality of the MLE

- $(u_i v_i)_{i\geq 1}$: stationary distribution of $\{Z_n^{\uparrow}\}$
- $\sigma^{2\uparrow}(z) = \dfrac{\sum_{k=1}^{\infty} k^2 Q_{zk}^{\uparrow}}{z^2} - (m^{\uparrow}(z))^2$

Theorem (Braunsteins, Hautphenne, M., 2022a)

Under (A1)–(A3), *for any* $z \in \mathbb{N}$, *initial state* i, *and* $x \in \mathbb{R}$, $\hat{m}_n(z)$ *satisfies*

$$\lim_{n\to\infty} \mathbb{P}_i[\{n/\gamma(z)\}^{1/2}\left(\hat{m}_n(z) - m^{\uparrow}(z)\right) \leq x \,|\, Z_n > 0] = \Phi(x),$$

where $\Phi(x)$ *is the standard normal distribution, and*

$$\gamma(z) := \frac{\sigma^{2\uparrow}(z)}{u_z v_z}.$$

Proof approach: coupling techniques and martingale CLT.

Asymptotic normality of the MLE

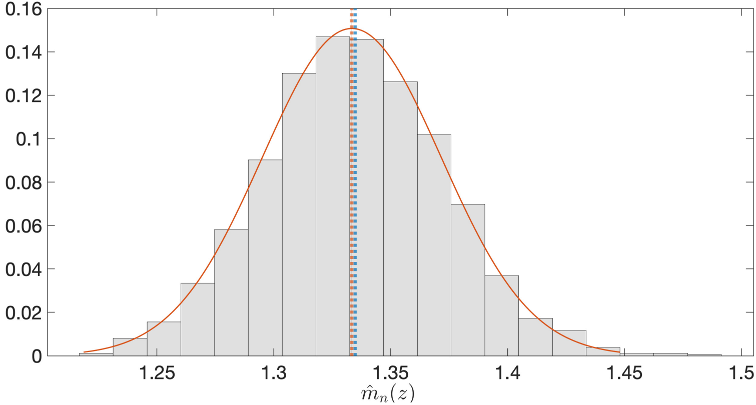

Fig: Histogram of $\hat{m}_n(z)$ for $z = 10$ and $n = 2000$, based on 5000 simulations, $K = 20$.

'Q-consistency' versus C-consistency

- 'Q-consistency': for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}[|\hat{m}_n(z) - m^{\uparrow}(z)| > \varepsilon \mid Z_n > 0] = 0$$

- C-consistency: for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}[|\tilde{m}_n(z) - m(z)| > \varepsilon \mid Z_n > 0] = 0$$

The estimator $\hat{m}_n(z)$ is Q-consistent but not C-consistent

- In our PSDBP, $m^{\uparrow}(z) \approx m(z)$ because $\{Z_\ell^{\uparrow}\} \approx \{Z_\ell\}$.

- The properties of the estimator $\hat{m}_n(z)$ are the key to obtain C-consistent estimators for the parameter θ_0.

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- The properties of the estimator $\hat{m}_n(z)$ are the key to obtain C-consistent estimators for the parameter $\boldsymbol{\theta}_0$.

C-consistent estimators for $\boldsymbol{\theta}_0$

We propose the following weighted least squares estimator of the

$$\hat{\boldsymbol{\theta}}_n = \arg\min_{\boldsymbol{\theta}\in\Theta} \sum_{z=1}^{\infty} \hat{w}_n(z) \left[\hat{m}_n(z) - m^{\uparrow}(z,\boldsymbol{\theta})\right]^2,$$

where the weights $\{\hat{w}_n(z)\}_{z\geq 1}$ are computed from the observations $Z_0, Z_1, \ldots, Z_n$, and are assumed to form an empirical distribution such that for any $z, i \geq 1$ and $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}_i[|\hat{w}_n(z) - w_z| > \varepsilon \mid Z_n > 0] = 0,$$

for some limiting distribution $\{w_z := w_z(\boldsymbol{\theta}_0)\}_{z\geq 1}$.

C-consistent estimators for $\boldsymbol{\theta}_0$

Under additional regularity assumptions on the parametric family

$$\mathcal{F}_\Theta = \{m^{\uparrow}(z,\boldsymbol{\theta}) : \boldsymbol{\theta} \in \Theta, z \in \mathbb{N}\},$$

we proved that $\hat{\boldsymbol{\theta}}_n$ is C-consistent and asymptotically normal.

Theorem (Braunsteins, Hautphenne, M., 2022b)

For any initial state i, and $\varepsilon > 0$, $\hat{\boldsymbol{\theta}}_n$ satisfies

$$\lim_{n\to\infty} \mathbb{P}_i\left[\left\|\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0\right\| > \epsilon | Z_n > 0\right] = 0 \qquad C\text{-consistency.}$$

C-consistent estimators for $\boldsymbol{\theta}_0$

Theorem (Braunsteins, Hautphenne, M., 2022b)

For any initial state i, and for any $\boldsymbol{x} = (x_1, \ldots, x_b) \in \mathbb{R}^b$,

$$\lim_{n\to\infty} \mathbb{P}_i \left[\sqrt{n} \left(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0 \right) \in (-\infty, x_1] \times \ldots \times (-\infty, x_b] \,|Z_n > 0 \right] = \Psi_{\beta(\boldsymbol{\theta}_0)}(\boldsymbol{x}),$$

where $\Psi_{\beta(\boldsymbol{\theta}_0)}(\cdot)$ is the distribution function of a b-dimensional normal r.v. with mean vector $\mathbf{0}$ and positive semi-definite covariance matrix

$$\beta(\boldsymbol{\theta}_0) := \eta(\boldsymbol{\theta}_0)^{-1}\zeta(\boldsymbol{\theta}_0)\eta(\boldsymbol{\theta}_0)^{-1},$$

where $\eta(\boldsymbol{\theta}_0)$ and $\zeta(\boldsymbol{\theta}_0)$ are the b-dimensional matrices given by

$$\begin{aligned}
\eta(\boldsymbol{\theta}_0) &= 2\sum_{z=1}^{\infty} w_z(\boldsymbol{\theta}_0)\nabla m^{\uparrow}(z,\boldsymbol{\theta}_0)\nabla m^{\uparrow}(z,\boldsymbol{\theta}_0)^{\top}, \\
\zeta(\boldsymbol{\theta}_0) &= 4\sum_{z=1}^{\infty} w_z(\boldsymbol{\theta}_0)^2\gamma(z,\boldsymbol{\theta}_0)\nabla m^{\uparrow}(z,\boldsymbol{\theta}_0)\nabla m^{\uparrow}(z,\boldsymbol{\theta}_0)^{\top},
\end{aligned}$$

Weight functions

We consider different weight functions:

(1) The proportion of generations with population size z:

$$\hat{w}_n^{(1)}(z) = \frac{\sum_{i=0}^{n-1} \mathbb{1}_{\{Z_i=z\}}}{n}.$$

Lemma (Braunsteins, Hautphenne, M., 2022b)

For any $z \geq 1$ and $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}_i\left[\left|\hat{w}_n^{(1)}(z) - w_z^{(1)}\right| > \epsilon | Z_n > 0\right] = 0,$$

where

$$w_z^{(1)} = u_z v_z.$$

Conditionally on $Z_n > 0$, the weights $\hat{w}_n^{(1)}(z)$ converge to the stationary distribution of the Q-process.

Weight functions

(2) The proportion of individuals who are alive when the population size is z:

$$\hat{w}_n^{(2)}(z) = \frac{z \sum_{i=0}^{n-1} \mathbb{1}_{\{Z_i = z\}}}{\sum_{i=0}^{n-1} Z_i}.$$

Lemma (Braunsteins, Hautphenne, M., 2022b)

For any $z \geq 1$ and $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}_i \left[\left| \hat{w}_n^{(2)}(z) - w_z^{(2)} \right| > \epsilon \middle| Z_n > 0 \right] = 0,$$

where

$$w_z^{(2)} = \frac{z\, u_z v_z}{\sum_{k=1}^{\infty} k u_k v_k}.$$

Conditionally on $Z_n > 0$, the weights $\hat{w}_n^{(2)}(z)$ converge to the size-biased distribution of the stationary distribution of the Q-process.

Comparison with the classical least squares estimator

- Classical least squares estimators:

$$\widetilde{\boldsymbol{\theta}}_n^* := \arg\min_{\boldsymbol{\theta}\in\Theta} \sum_{k=1}^{n} w_k \left\{Z_k - Z_{k-1}\, m(Z_{k-1}, \boldsymbol{\theta})\right\}^2,$$

where $\{w_k\}$ is an appropriately chosen weighting function.

Proposition (Braunsteins, Hautphenne, M., 2022b)

The least squares estimator $\widehat{\boldsymbol{\theta}}_n$ with weight $\{w_z^{(1)}\}$ or $\{w_z^{(2)}\}$ is equal to the previous estimator modified such that $m(\cdot)$ is replaced by $m^{\uparrow}(\cdot)$,

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Outline

Simulated example: estimation in the quasi-stationary phase

We consider the PSDBP with a carrying capacity and binary fission reproduction given by a modified BH model:

$$p_2(z,K) = \frac{vK}{K+(2v-1)z}, \quad p_0(z,K) = 1 - p_2(z,K), \quad z \in \{2i : i \in \mathbb{N}\}.$$

The offspring parameter is $\boldsymbol{\theta} = (K, v) \in \Theta = (0,\infty) \times (1/2, 1]$.

We fixed $\boldsymbol{\theta}_0 = (K_0, v_0) = (50, 0.70)$.

We simulated 2000 non-extinct trajectories, and we are interested in estimating the parameters based on the observation of $Z_0, Z_1, \ldots, Z_{2000}$.

Estimation in the quasi-stationary phase

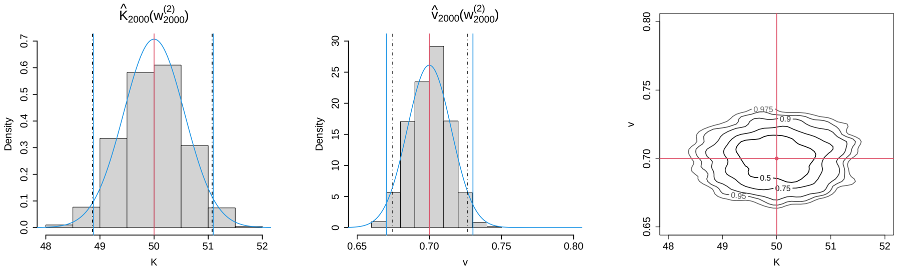

Fig: Left: marginal distribution of K, together with the empirical and theoretical marginal 95% confidence intervals. Centre: marginal distribution of the estimator of v, together with the empirical and theoretical marginal 95% confidence intervals. Right: confidence regions for levels $50\%, 75\%, 90\%, 95\%, 97.5\%$.

Simulated example: estimation in the growth phase

We consider the PSDBP with a carrying capacity and binary fission reproduction given by a modified BH model:

$$p_2(z,K)=\frac{vK}{K+(2v-1)z},\quad p_0(z,K)=1-p_2(z,K),\quad z\in\{2i:i\in\mathbb{N}\}.$$

The offspring parameter is $\boldsymbol{\theta}=(K,v)\in\Theta=(0,\infty)\times(1/2,1]$.

We fixed $\boldsymbol{\theta}_0=(K_0,v_0)=(200,0.75)$.

We simulated 1000 non-extinct trajectories, and we are interested in estimating the parameters based on the observation of $Z_0,Z_1,\ldots,Z_{12}$.

Estimation in the growth phase

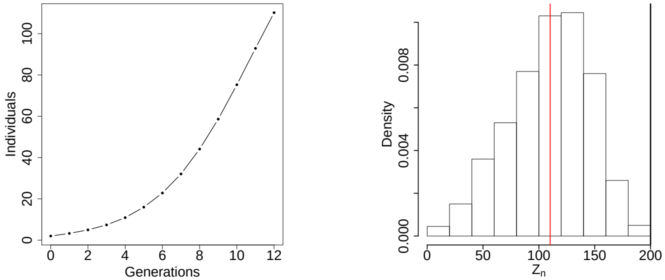

Fig: **Left:** empirical mean of the simulated paths. **Right:** empirical distribution of Z_{12}, together with the empirical mean (red line) and the carrying capacity (grey line).

Simulated example: estimation in the growth phase

$$\hat{\boldsymbol{\theta}}_n = \arg\min_{\boldsymbol{\theta}\in\Theta} \sum_{z=1}^{\infty} \hat{w}_n(z) \left\{ \hat{m}_n(z) - m^{\uparrow}(z, \boldsymbol{\theta}) \right\}^2 \qquad (C\text{-consistent})$$

$$\tilde{\boldsymbol{\theta}}_n^* = \arg\min_{\boldsymbol{\theta}\in\Theta} \sum_{z=1}^{\infty} \hat{w}_n(z) \left\{ \hat{m}_n(z) - m(z, \boldsymbol{\theta}) \right\}^2 \qquad (\text{modified})$$

(1) $\hat{w}_n^{(1)}(z) = \sum_{i=0}^{n-1} \mathbb{1}_{\{Z_i = z\}} / n$

(2) $\hat{w}_n^{(2)}(z) = z \sum_{i=0}^{n-1} \mathbb{1}_{\{Z_i = z\}} / (\sum_{i=0}^{n-1} Z_i)$

Table: Median of the 1000 estimates — $\boldsymbol{\theta}_0 = (K_0, v_0) = (200, 0.75)$

	C-consistent		Modified	
	$\hat{K}_n^*$	$\hat{v}_n^*$	$\tilde{K}_n^*$	$\tilde{v}_n^*$
(1)	177.60010	0.77088	151.31210	0.79670
(2)	191.78720	0.76622	185.79590	0.77025

Carrying capacity of the black robin population

Model	$\hat{K}$	$\hat{v}$
Beverton-Holt	109.61	0.696
Ricker	95.64	0.679

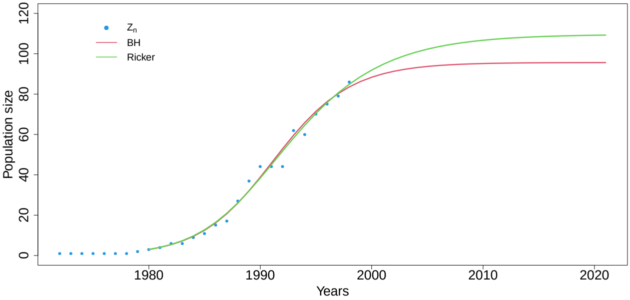

Fig: Number of adult females between 1972 and 1998, together with the mean population size curve of the estimated BH and Ricker models.

Outline

Conclusions

- We obtained the MLEs of the offspring mean function of a PSDBP based on the observation of the population sizes.
- Focussing our attention on the study of PSDBPs whose extinction is certain, we considered the sample of the population sizes and we analysed the asymptotic properties of the MLE $\hat{m}_n(z)$ for a population size z fixed, establishing its Q-consistency and asymptotic normality.
- In a parametric setting, we developed C-consistent estimators for the offspring parameter θ_0 of PSDBPs.
- We applied our results to estimate the carrying capacity of the endangered black robin population in the Chatham Islands.
- The choice of weights and parametric model are still very important.

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Our assumptions

(A1) If $m^{\uparrow}(z, \boldsymbol{\theta}_1) = m^{\uparrow}(z, \boldsymbol{\theta}_2)$, for each $z \in \text{supp}(\boldsymbol{w}(\boldsymbol{\theta}_0))$, then $\boldsymbol{\theta}_1 = \boldsymbol{\theta}_2$

(A2) $M = \sup_{\boldsymbol{\theta}\in\Theta} \sup_{z\in\mathbb{N}} m^{\uparrow}(z, \boldsymbol{\theta}) < \infty$.

(A3) For each $\epsilon > 0$,
$\lim_{n\to\infty} P_i[\sup_{z\in\mathbb{N}} |\hat{m}_n(z) - m^{\uparrow}(z, \boldsymbol{\theta}_0)| > \epsilon | Z_n > 0] = 0$, for any initial state $i \in \mathbb{N}$.

(A4) The function $m^{\uparrow}(z, \boldsymbol{\theta})$ is twice continuously differentiable with respect to $\boldsymbol{\theta}$ for each $z \in \mathbb{N}$. Moreover, for each $\boldsymbol{\theta}' \in \Theta$, there exists a compact set C such that $\boldsymbol{\theta}' \in int(C)$:

1. $M_1^*(\boldsymbol{\theta}') = \sup_{\boldsymbol{\theta}\in C} \sup_{z\in\mathbb{N}} \left\| \nabla m^{\uparrow}(z, \boldsymbol{\theta}) \right\|_{\infty} < \infty$.
2. $M_2^*(\boldsymbol{\theta}') = \sup_{\boldsymbol{\theta}\in C} \sup_{z\in\mathbb{N}} \max_{i,j=1,\dots,d} \left| \frac{\partial^2 m^{\uparrow}(z,\boldsymbol{\theta})}{\partial\theta_i \partial\theta_j} \right| < \infty$.

Ingredient for the proof of C-consistency

We use the following lemma:

Lemma

Let $f : \boldsymbol{\theta} \in \Theta \mapsto [0, \infty)$ be a continuous function with a unique minimum at $\boldsymbol{\theta}_0$ that satisfies $f(\boldsymbol{\theta}_0) = 0$.

Let $\hat{f}_n(\boldsymbol{\theta})$ be a (random) measurable function of the random variables $Z_0, \ldots, Z_n$ for each $\boldsymbol{\theta} \in \Theta$, s.t. for each $\omega \in \Omega$, $\hat{f}_n(\omega, \cdot)$ is a continuous function on Θ.

$$\text{If} \quad \forall \epsilon > 0, \quad \lim_{n\to\infty} P\left[\sup_{\boldsymbol{\theta}\in\Theta} |\hat{f}_n(\boldsymbol{\theta}) - f(\boldsymbol{\theta})| > \epsilon | Z_n > 0\right] = 0, \tag{1}$$

then

1. $\lim_{n\to\infty} P\left[\exists \arg\min_{\boldsymbol{\theta}\in\Theta} \hat{f}_n(\boldsymbol{\theta}) | Z_n > 0\right] = 1.$
2. $\forall \epsilon > 0$, $\lim_{n\to\infty} P\left[\left\| \arg\min_{\boldsymbol{\theta}\in\Theta} \hat{f}_n(\boldsymbol{\theta}) - \boldsymbol{\theta}_0 \right\| > \epsilon | Z_n > 0\right] = 0.$

Idea of the proof of C-consistency

We apply the previous lemma with

$$\hat{f}_n(\boldsymbol{\theta}) = \sum_{z=1}^{\infty} \hat{w}_n(z) \left\{ \hat{m}_n(z) - m^{\uparrow}(z, \boldsymbol{\theta}) \right\}^2$$

and

$$f(\boldsymbol{\theta}) = \sum_{z=1}^{\infty} w_z(\boldsymbol{\theta}_0) \left\{ m^{\uparrow}(z, \boldsymbol{\theta}_0) - m^{\uparrow}(z, \boldsymbol{\theta}) \right\}^2$$

and prove that (1) holds. To that end we introduce

$$\tilde{f}_n(\boldsymbol{\theta}) = \sum_{z=1}^{\infty} w_z(\boldsymbol{\theta}_0) \mathbb{1}_{\{j_n(z)>0\}} \left\{ \hat{m}_n(z) - m^{\uparrow}(z, \boldsymbol{\theta}) \right\}^2, \quad n \in \mathbb{N},\ \boldsymbol{\theta} \in \Theta,$$

and we prove (using our assumptions) that for each $\epsilon > 0$,

1. $\lim_{n\to\infty} P\left[\sup_{\boldsymbol{\theta}\in\Theta} \left|\hat{f}_n(\boldsymbol{\theta}) - \tilde{f}_n(\boldsymbol{\theta})\right| > \epsilon | Z_n > 0\right] = 0.$
2. $\lim_{n\to\infty} P\left[\sup_{\boldsymbol{\theta}\in\Theta} \left|\tilde{f}_n(\boldsymbol{\theta}) - f(\boldsymbol{\theta})\right| > \epsilon | Z_n > 0\right] = 0.$

Idea of the proof of asymptotic normality

We use a Taylor expansion for the function $\nabla \hat{f}_n(\cdot)$ around $\boldsymbol{\theta}_0$ on the set $\{Z_n > 0\}$,

$$\mathbf{0} = \nabla \hat{f}_n(\hat{\boldsymbol{\theta}}_n) = \nabla \hat{f}_n(\boldsymbol{\theta}_0) + \nabla^2 \hat{f}_n(\boldsymbol{\theta}_n)^\top (\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0),$$

where $\boldsymbol{\theta}_n$ is a point between $\hat{\boldsymbol{\theta}}_n$ and $\boldsymbol{\theta}_0$, and $\nabla^2 \hat{f}_n(\boldsymbol{\theta}_n)$ is the Jacobian matrix of $\hat{f}_n(\cdot)$ at $\boldsymbol{\theta}_n$. Then,

$$\sqrt{n}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) = -\left(\nabla^2 \hat{f}_n(\boldsymbol{\theta}_n)\right)^{-1} \sqrt{n}\, \nabla \hat{f}_n(\boldsymbol{\theta}_0).$$

Idea of the proof of asymptotic normality

We then prove the result in two steps:

1. If $\Psi_{\zeta(\boldsymbol{\theta}_0)}(\cdot)$ is the distribution function of a b-dimensional normal distribution with mean vector $\mathbf{0}$ and covariance matrix $\zeta(\boldsymbol{\theta}_0)$, and $x_1, \ldots, x_b \in \mathbb{R}$, then

$$\lim_{n\to\infty} P\left[-\sqrt{n}\,\frac{\partial \hat{f}_n(\boldsymbol{\theta}_0)}{\partial\theta_j} \leq x_j,\ j=1,\ldots,b \,\middle|\, Z_n > 0\right] = \Psi_{\zeta(\boldsymbol{\theta}_0)}(x_1,\ldots,x_b).$$

2. For each $\epsilon > 0$,

$$\lim_{n\to\infty} P\left[\left|\frac{\partial^2 \hat{f}_n(\boldsymbol{\theta}_n)}{\partial\theta_j\partial\theta_l} - \eta_{jl}(\boldsymbol{\theta}_0)\right| > \epsilon | Z_n > 0\right] = 0, \quad j,l=1,\ldots,b.$$

Estimation in the quasi-stationary phase

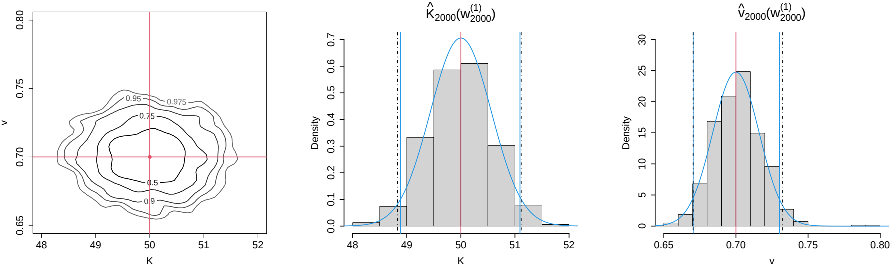

Fig: Left: confidence regions for levels 50%, 75%, 90%, 95%, 97.5%. Centre: marginal distribution of K. Right: marginal distribution of the estimator of v.