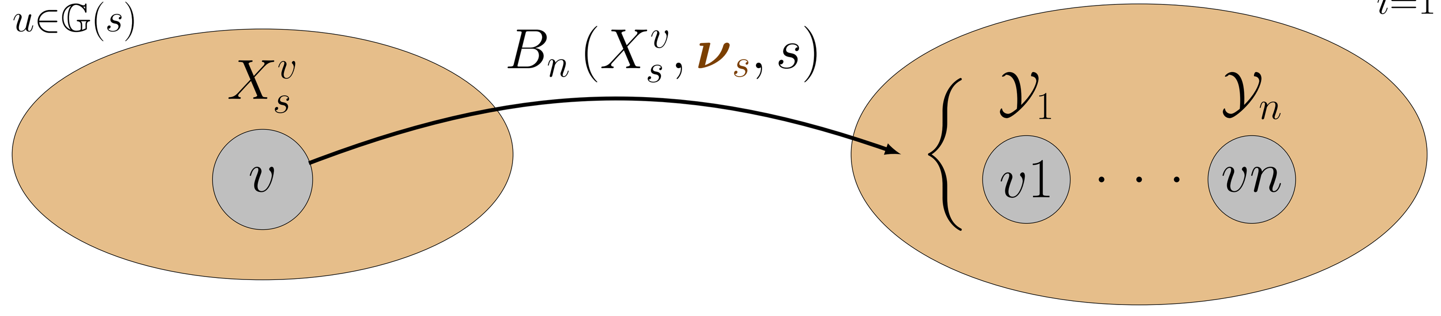
Spinal construction for interacting branching processes and applications

Charles Medous

A general interacting population

We denote $\mathbb{G}(s)$ the set of individuals living in the population $\boldsymbol{\nu}_s$ at time s, where every individual has a label u in $\mathcal{U}$ and a trait X_s^u in $\mathcal{X} \subset \mathbb{R}^d$.

$$\boldsymbol{\nu}_s = \sum_{u \in \mathbb{G}(s)} \delta_{X_s^u} \qquad \boldsymbol{\nu}_{s+} = \boldsymbol{\nu}_s - \delta_{X_s^v} + \sum_{i=1}^{n} \delta_{\mathcal{Y}_i}$$



The traits at birth $(\mathcal{Y}_i)_{i \le n}$ of the children are r.v. distributed according to

$$(\mathcal{Y}_1, \cdots, \mathcal{Y}_n) \sim K_n(X_s^v, \boldsymbol{\nu}_s, s, \cdot).$$

During its life, at all time $t \ge s$, the trait of the individual vi is given by

$$X_t^{vi} = \mathcal{Y}_i \exp\left(\int_s^t \mu(X_r^{vi}, \boldsymbol{\nu}_r, r)\, \mathrm{d}r\right).$$

Defining the ψ_ϕ-spine process

We construct a spine process $(\boldsymbol{\chi}_t)_{t\ge0}$ with a weight function $\psi_\phi(x, \boldsymbol{\chi}) := \psi(x, \langle \boldsymbol{\chi}, \phi \rangle)$. We denote $\boldsymbol{\chi}_+(x, \mathbf{y}) := \boldsymbol{\chi} - \delta_x + \sum_{i=1}^n y_i$ the population after the branching of an individual of trait x to children of traits given by $\mathbf{y}$.

- The spinal individual of trait x_e branches to children of traits $\mathbf{y}$ at rate
$$\widehat{\Gamma}_n^*(x_e, \boldsymbol{\chi}, t, \mathrm{d}\mathbf{y}) := B_n(x_e, \boldsymbol{\chi}, t) \frac{\sum_{i=1}^n \psi_\phi(y_i, \boldsymbol{\chi}_+(x_e, \mathbf{y}))}{\psi_\phi(x_e, \boldsymbol{\chi})} K_n(x_e, \boldsymbol{\chi}, t, \mathrm{d}\mathbf{y}).$$
For all $1 \le j \le n$ the j-th child is chosen to be the new spine with probability
$$\frac{\psi_\phi(y_j, \boldsymbol{\chi}_+(x_e, \mathbf{y}))}{\sum_{i=1}^n \psi_\phi(y_i, \boldsymbol{\chi}_+(x_e, \mathbf{y}))}.$$
- For the individuals outside the spine, the branching rate is
$$\widehat{\Gamma}_n(x_e, x, \boldsymbol{\chi}, t, \mathrm{d}\mathbf{y}) := B_n(x_e, \boldsymbol{\chi}, t) \frac{\psi_\phi(x_e, \boldsymbol{\chi}_+(x, \mathbf{y}))}{\psi_\phi(x_e, \boldsymbol{\chi})} K_n(x, \boldsymbol{\chi}, t, \mathrm{d}\mathbf{y}).$$

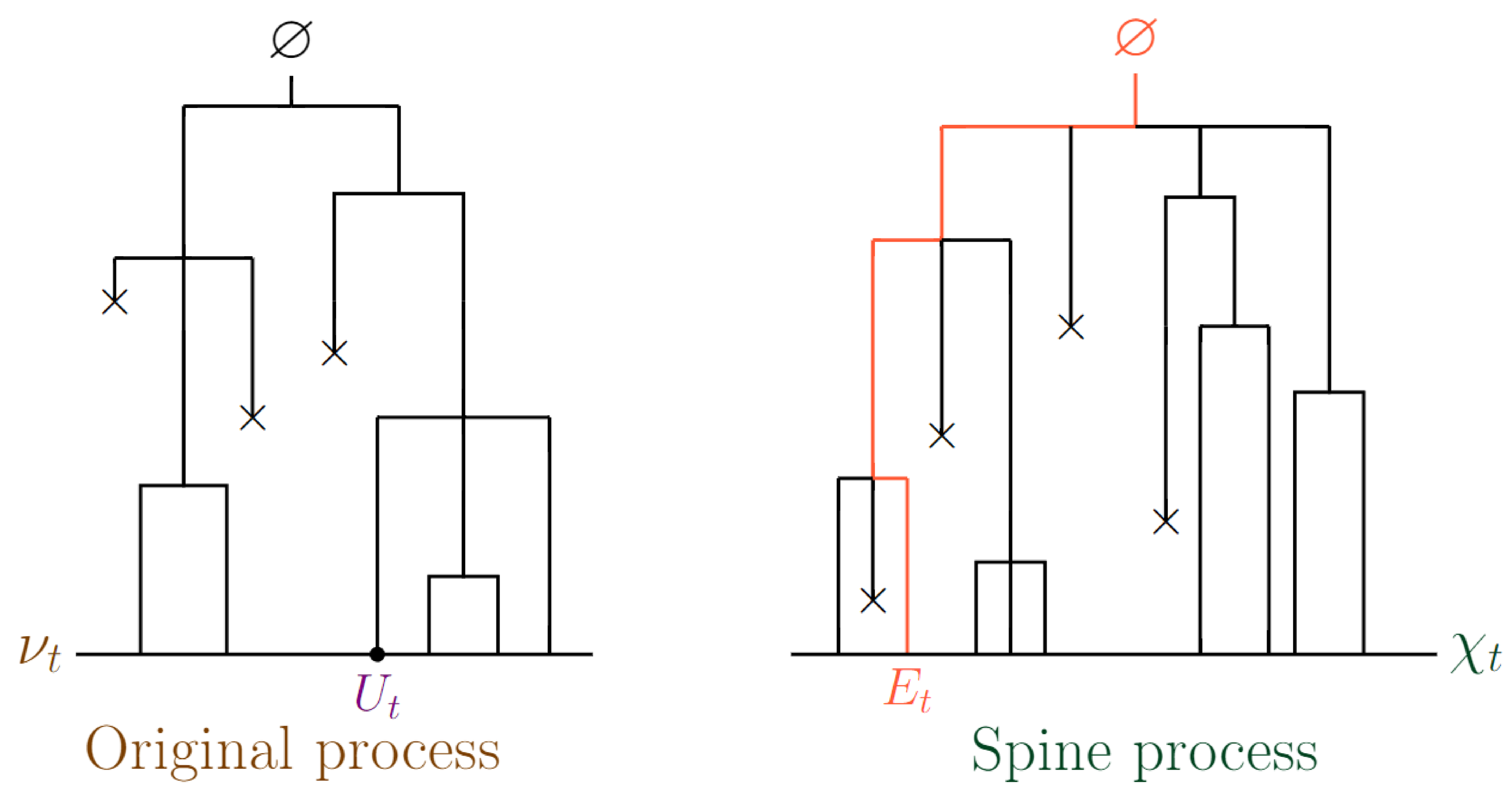
Girsanov type result for the spinal construction

Let ψ_ϕ be an "good" weight function and $(E_t, \boldsymbol{\chi}_t)_{t\ge0}$ the associated spine process with spine E_t of trait Y_t. Let U_t be a random variable sampling uniformly an individual in the population at time t. For every initial measure z and every measurable non-negative function H,

$$\mathbb{E}_z\left[\mathbb{1}_{\{T_{Exp}>t, \mathbb{G}(t)\neq\emptyset\}} H(U_t, (\boldsymbol{\nu}_s, s \le t))\right] = \langle z, \psi_\phi(\cdot, z)\rangle \mathbb{E}_z\left[\mathbb{1}_{\{\widehat{T}_{Exp}>t\}} \exp\left(\int_0^t \frac{\boldsymbol{\mathcal{G}}^s \boldsymbol{\psi_\phi}(Y_s, \boldsymbol{\chi}_s)}{\boldsymbol{\psi_\phi}(Y_s, \boldsymbol{\chi}_s)} \mathrm{d}s\right) \frac{H(E_t, (\boldsymbol{\chi}_s, s \le t))}{\langle \boldsymbol{\chi}_t, 1\rangle \psi_\phi(Y_t, \boldsymbol{\chi}_t)}\right], \qquad (1)$$

where for all $s \ge 0$, noting $(\mathcal{A}^s)_{s\ge0}$ the generators of the original process,

$$\frac{\boldsymbol{\mathcal{G}}^s \boldsymbol{\psi_\phi}(Y_s, \boldsymbol{\chi}_s)}{\boldsymbol{\psi_\phi}(Y_s, \boldsymbol{\chi}_s)} := \frac{\mathcal{A}^s \psi_\phi(Y_s, \boldsymbol{\chi}_s)}{\psi_\phi(Y_s, \boldsymbol{\chi}_s)} + \sum_{n\ge0}\left[\int_{\mathcal{X}^n} \widehat{\Gamma}_n^*(Y_s, \boldsymbol{\chi}_s, s, \mathrm{d}\mathbf{y}) - \int_{\mathcal{X}^n} \widehat{\Gamma}_n(Y_s, Y_s, \boldsymbol{\chi}_s, s, \mathrm{d}\mathbf{y})\right].$$



This result characterizes the joint probability distribution of $(U_t, (\boldsymbol{\nu}_s, s \le t))$, that is the randomly sampled individual in the population $\boldsymbol{\nu}_t$ at time t , along with the entire population trajectory up to that time.
Equation (1) provides a Girsanov type change of measure that connects this pair to the law $(E_t, (\boldsymbol{\chi}_s, s \le t))$ of the spine process with its distinguished individual.

A Kesten-Stigum type result

Using (1) with some non-explosion assumptions, we show that for every ψ_ϕ,

$$W_t(\psi_\phi) := \sum_{u \in \mathbb{G}(t)} \exp\left(-\int_0^t \frac{\boldsymbol{\mathcal{G}}^t \boldsymbol{\psi_\phi}(X_s^u, \boldsymbol{\nu}_s)}{\boldsymbol{\psi_\phi}(X_s^u, \boldsymbol{\nu}_s)} \mathrm{d}s\right) \psi_\phi(X_t^u, \boldsymbol{\nu}_s)$$

is a non-negative martingale with respect to the canonical filtration of the process $(\boldsymbol{\nu}_t)_{t\ge0}$.
We assume that there exist $c, C \in \mathbb{R}_+^*$ such that, for all (x, ν, t)

$$c \le \sum_{n\ge0} B_n(x, \nu, t)(n-1) \le C.$$

Using the spinal decomposition method we obtain the following result:

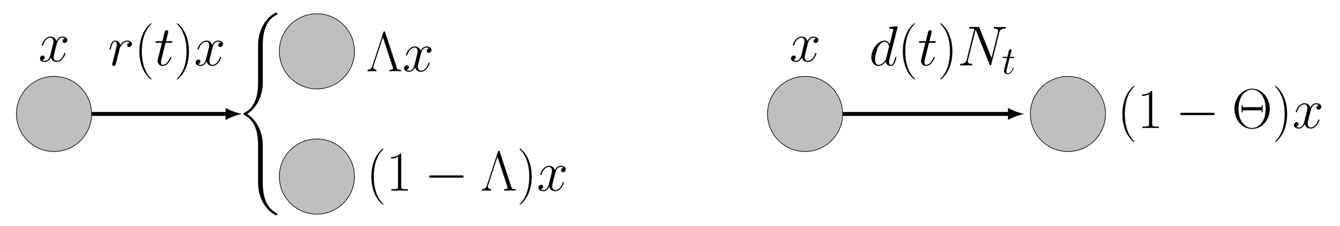
- If $\sum_{k\ge1} k\log(k) \sup_{x,\nu,t} B_k(x,\nu,t) < +\infty$ then, for all initial measure z
$$\mathbb{E}_z[\limsup W_t(1)] = \langle z, 1\rangle.$$
- If $\sum_{k\ge1} k\log(k) \inf_{x,\nu,t} B_k(x,\nu,t) = +\infty$, then
$$\limsup W_t(1) = 0 \quad \text{almost surely.}$$

References

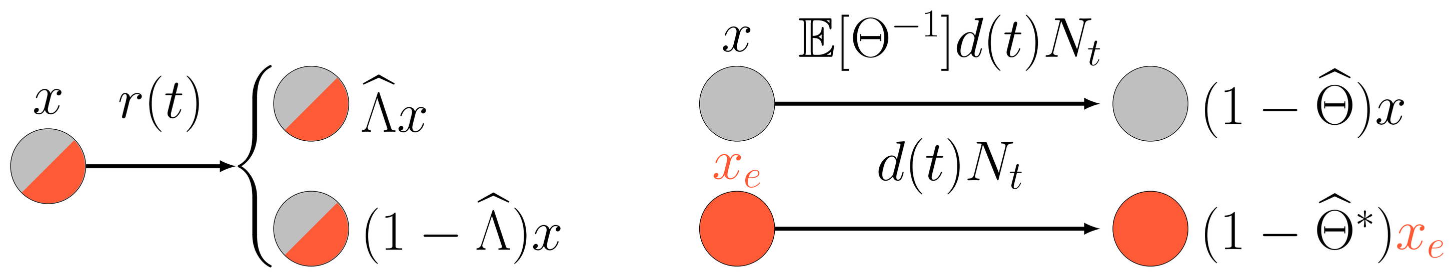
[1] R. Lyons, R. Pemantle, and Y. Peres. Conceptual proofs of $L \log L$ criteria for mean behavior of branching processes. The Annals of Probability, pages 1125–1138, 1995.
[2] K. B. Athreya. Change of measures for Markov chains and the $L \log L$ theorem for branching processes. Bernoulli, pages 323–338, 2000.
[3] V. Bansaye. Spine for interacting populations and sampling. ArXiv preprint arXiv:2105.03185, 2022

A Yule process with competitive interactions

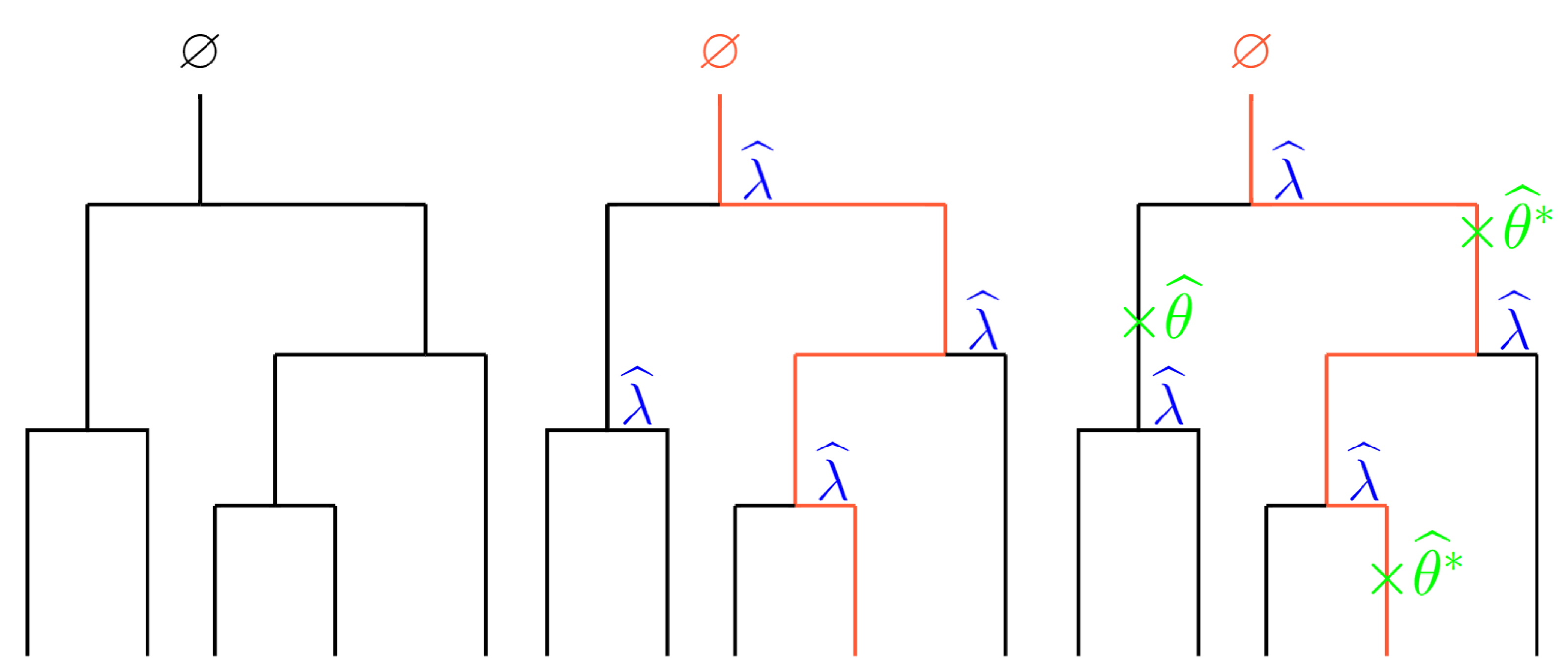
Every individual is characterized by its mass $x \in \mathbb{R}_+^*$, that grows according to $\dot{x} = \mu(t)x$. The branching events are



Choosing $\psi(x,y) := xe^{-y}$ and $\phi(x) := \ln\left(x\mathbb{E}[(\Lambda(1-\Lambda))^{-1}]\right)$, we have for the spine process



Algorithm draft



(a) Step 1 (b) Step 2 (c) Step 3