Coming down from infinity and Explosion in Exchangeable Fragmentation-Coalescence Processes

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Imagine a collection of objects that merge and fragmentate randomly along the time:

- If we start from infinitely many objects, is the coalescence strong enough for having finitely many ones at some time? $\rightarrow$ *coming down from infinity*
- If we start from a finite number of objects, is the fragmentation strong enough for having infinitely many ones at some time? $\rightarrow$ *explosion*
- Can we find regimes where the configuration with infinitely many objects is *regular*?

We study the setting of Exchangeable Fragmentation-Coalescence processes.

Exchangeable fragmentation-coalescence (EFC)

An EFC process is a Markov process $(\Pi(t), t \geq 0)$ valued in the space of partitions of $\mathbb{N}$:

$$\mathcal{P}_\infty := \{\pi = (\pi_1, \pi_2, \cdots); \cup_{i\geq 1}\pi_i = \mathbb{N}\},$$

endowed with the distance $d(\pi,\pi') = \max\{n \geq 1 : \pi_{|[n]} = \pi'_{|[n]}\}^{-1}$, with càdlàg paths, such that

- for all $t \geq 0$, $\Pi(t)$ is an exchangeable partition, i.e

$$\sigma\Pi(t) \overset{\mathcal{L}}{=} \Pi(t) \quad \forall \sigma \text{ permutation with finite support.}$$

- the process evolves by coalescence of blocks or fragmentation of one block.

Characterisation & construction[1] (J. Berestycki (2004)): Any EFC is characterised in law by two σ-finite *exchangeable* measures μ_{Coag} and μ_{Frag} on $\mathcal{P}_\infty$.

[1] by compatibility through restriction

Exchangeability's consequences and Poisson construction

Facts

For any exchangeable random partition π

- $\forall i \geq 1$, *if* $\pi_i \neq \emptyset$ *then it is either infinite or a singleton.*
- *There are either infinitely many singletons (dust) or none.*

Sketch of construction: coalescent part:

Let $\mathrm{PPP}_C = \sum_{t>0} \delta_{(t,\pi^c)}$ be a Poisson Point Process (PPP) on $\mathcal{P}_\infty$ with intensity $\mathrm{d}t \times \mathrm{d}\mu_{\mathrm{Coag}}$. If (t, π^c) is an atom of PPP_C:

$$\Pi(t) = \mathrm{Coag}(\Pi(t-), \pi^c) := \{\cup_{\ell \in \pi^c_i} \Pi_\ell(t-), i \geq 1\}.$$

For instance, if

- $\Pi(t-) = \{\{1, 2, \cdots\}, \{3, 5, \cdots\}, \{4, 6, \cdots\}, \{\cdots\}, \cdots\}$
- $\pi^c = \{\{1\}, \{2, 3, 4, \cdots\}, \{5\}, \cdots\}$

then

$$\Pi(t) = \{\{1, 2, \cdots\}, \{3, 4, 5, 6, \cdots, \cdots, \cdots, \cdots\}, \cdots\}.$$

Sketch of construction: fragmentation part:

Let $\mathrm{PPP}_F = \sum_{t>0} \delta_{(t,\pi^f,j)}$ be an indep PPP$(\mathrm{d}t \times \mathrm{d}\mu_{\mathrm{Frag}} \times \mathrm{d}\#)$ with $\#$ the counting measure.
If (t, π^f, j) is an atom of PPP_F:

$$\Pi(t) = \mathrm{Frag}(\Pi(t-), \pi^f, j) := \{\Pi_\ell(t-), \ell \neq j, \Pi_j(t-) \cap \pi_i^f, i \geq 1\}.$$

For instance, if

- $\Pi(t-) = \{\{1, 2, \cdots\}, \{3, 4, 5, 6, \cdots\}, \cdots\}$
- $j = 2$ and $\pi^f = \{\{1, 2, 4, 5 \cdots\}, \{3, 6, \cdots\}, \cdots\}$

then

$$\Pi(t) = \{\{1, 2, \cdots\}, \{3, 6, \cdots\}, \{4, 5, \cdots\}, \cdots\}.$$

Denote by $(\#\Pi(t), t \geq 0)$ the process of the number of blocks.

Question (Coming down from infinity)

Assume $\#\Pi(0) = \infty$*:* $\exists t > 0; \#\Pi(t) < \infty$ *?*

Pitman-Schweinsberg's zero-one law: if no coalescence into finitely many blocks at once is allowed then, setting $\tau_\infty := \inf\{t > 0 : \#\Pi(t) < \infty\}$,

$$\mathbb{P}(\tau_\infty = 0) = 1 \text{ or } \mathbb{P}(\tau_\infty = \infty) = 1.$$

Pure Coalescents: $\mu_{\mathrm{Frag}} \equiv 0$.

1. Only sufficient conditions are known in the general case (Ξ-coalescents: multiple simultaneous mergings) Schweinsberg (EJP 2003), Herriger and Möhle (ALEA 2012).
2. Schweinsberg found a **necessary and sufficient condition** for the Λ-coalescents (no simultaneous multiple mergings) for which μ_{Coag} is carried over simple partitions, i.e. those with **only one non-singleton block**.

Some EFCs:

1. J. Berestycki showed that if $\mu_{\mathrm{Frag}}(\mathcal{P}_\infty) = \infty$, $\#\Pi(t) = \infty$ apart perhaps at exceptional times when all blocks coalesce instantaneously into finitely many.
2. 'Fast'-EFC: Kyprianou, Pagett, Rogers, Schweinsberg (AOP2017) studied Kingman coalescence versus fragmentation of a block into singletons:
   - $\mu_{\mathrm{Coag}} = c_{\mathrm{K}} \sum_{i<j} \delta_{K(i,j)}$ with $K(i,j) := (\cdots, \{i,j\}, \cdots)$ where here $\cdots$ are singletons and $c_{\mathrm{K}} > 0$
   - $\mu_{\mathrm{Frag}} = \lambda \delta_{0_{[\infty]}}$, where $0_{[\infty]} = \{\{1\}, \{2\}, \cdots\}$ and $\lambda \geq 0$.

$\rightarrow$ $(\Pi(t), t \geq 0)$ comes down from infinity iff $2\lambda/c_{\mathrm{K}} < 1$.

When $0 < 2\lambda/c_{\mathrm{K}} < 1$, the process $\#\Pi$ makes excursions away from ∞, and the boundary ∞ is regular for itself.

Simple EFC processes

A simple EFC is a process $(\Pi(t), t \geq 0)$ such that

1. coalescences are multiple but not simultaneous:

   Λ-coalescent.

2. The fragmentations have finite rate:

$$\mu_{\mathrm{Frag}}(\mathcal{P}_\infty) < \infty$$

and there is no **fragmentations into singletons**:

$$\mu_{\mathrm{Frag}}(\{\pi : \pi \text{ contains singletons }\}) = 0$$

Facts

- *The process $\#\Pi$ is right-continuous, and at any time t such that $\#(\Pi(t-)) < \infty$, it has left-limits.*
- *When the simple EFC process Π evolves in the space of finite partitions, $\mathcal{P}_\infty^{\mathrm{finite}} := \{\pi \in \mathcal{P}_\infty : \#\pi < \infty\}$, the process $\#\Pi$ is Markov with piecewise constant sample paths in $\mathbb{N}$.*

Proposition

Let $(\Pi(t), t \geq 0)$ *be a simple EFC:*

$$(\#\Pi(t), t < \zeta) \text{ started from } \#\Pi(0) = n \in \mathbb{N}$$

with $\zeta := \inf\{t > 0; \#\Pi(t-) \text{ or } \#\Pi(t) = \infty\}$, *is a Markov process with generator* $\mathcal{L} = \mathcal{L}^c + \mathcal{L}^f$ *defined by*

$$\mathcal{L}^c g(n) := \sum_{2 \leq k \leq n} \binom{n}{k} \lambda_{n,k} \left(g(n-k+1) - g(n)\right)$$

$$\text{with } \lambda_{n,k} := \int_{]0,1]} x^k (1-x)^{n-k} x^{-2} \Lambda(\mathrm{d}x) + c_{\mathrm{K}} \mathbb{1}_{\{k=2\}}.$$

$$\mathcal{L}^f g(n) := \sum_{1 \leq k \leq \infty} n\mu(k) \left(g(n+k) - g(n)\right).$$

- $\Lambda(\mathrm{d}x) := x^2 \mu_{\mathrm{Coag}}(|\pi|^{\downarrow} \in \mathrm{d}x)$ **coalescence measure,**
- $\mu(k) := \mu_{\mathrm{Frag}}(\#\pi = k-1)$ **splitting measure,**

Coming down from infinity of Λ-coalescents

Set for all $n \geq 2$

$$\Phi(n) := \sum_{k=2}^{n} \binom{n}{k} \lambda_{n,k}(k-1)$$
$$= \int_{]0,1[} ((1-x)^n + nx - 1)x^{-2}\Lambda(\mathrm{d}x) + c_{\mathrm{K}} \binom{n}{2}$$
$$= \text{total rate of decrease of } \#\Pi \text{ starting from } n \text{ blocks.}$$

Schweinsberg's condition for coming down from infinity (CDI): Let Π be a Λ-coalescent. If $\Lambda(\{1\}) = 0$ (no instantaneous total coalescence allowed) and $\#\Pi(0) = \infty$ then

$$\#\Pi(t) < \infty \text{ for all } t > 0 \text{ a.s.} \iff \sum_{n=2}^{\infty} \frac{1}{\Phi(n)} < \infty. \qquad (1)$$

Coming down from infinity of simple EFCs

Let Π be a simple EFC. We suppose $\Lambda(\{1\}) = 0$.
Set

$$\bar{\mu}(k) := \mu(\{k, \cdots, \infty\}) \text{ for all } k \geq 1.$$

Theorem (F. 2022)

Assume $\#\Pi(0) = \infty$ and $\sum_{n=2}^{\infty} \frac{1}{\Phi(n)} < \infty$. Let $\theta^\star$ and $\theta_\star$ in $[0, \infty]$ be:

$$\theta_\star := \liminf_{n\to\infty} \sum_{k=1}^{\infty} \frac{n\bar{\mu}(k)}{\Phi(k+n)} \text{ and } \theta^\star := \limsup_{n\to\infty} \sum_{k=1}^{\infty} \frac{n\bar{\mu}(k)}{\Phi(k+n)},$$

- *If $\theta^\star < 1$ then Π comes down from infinity.*
- *If $\theta_\star > 1$ then Π stays infinite.*

⚠ $\#\Pi$ might explode even if it comes down from infinity.

A heuristic

Consider the function $f : n \mapsto \sum_{j=n+1}^{\infty} \frac{1}{\Phi(j)}$.

1. $f(n) \approx$ time needed for the pure coalescent to go below level $n+1$ when started from ∞, (speed of coming down from infinity of the Λ-coalescent$= v_t := \inf\{u > 0 : f(u) > t\}$: Limic, Berestycki[2], AoP 2010).
2. Let Z be the nber of blocks formed by a fragmentation event: Z has law $\mu(\cdot)/\mu(\bar{\mathbb{N}})$ and the *mean arrival time* of a fragmentation is $1/n\mu(\bar{\mathbb{N}})$.
3. By Fubini,
$$\sum_{k=1}^{\infty} \frac{n\bar{\mu}(k)}{\Phi(n+k)} = n\mu(\bar{\mathbb{N}})\mathbb{E}\left[\sum_{k=n+1}^{n+Z} \frac{1}{\Phi(k)}\right] = \frac{\mathbb{E}[f(n) - f(n+Z)]}{1/\mu(\bar{\mathbb{N}})n}.$$

$\theta^\star < 1$ iff $\#\Pi$ jumps from n to $n+Z$ at smaller rate than it comes down from $n+Z$ to n for arbitrary large n.

Corollary (A moment condition)

Suppose $\mu(\infty) = 0$ *and* $\sum_{n=2}^{\infty} 1/\Phi(n) < \infty$.

$$\text{If } \sum_{n=2}^{\infty} \frac{n}{\Phi(n)} \bar{\mu}(n) < \infty, \text{ then } \theta = 0 \Longrightarrow \text{comes down from infinity.}$$

In particular if μ *has a first moment then* $\theta = 0$.

Corollary (Kingman coalescent versus fragmentation into infinitely many blocks)

Suppose $\sum_{n=2}^{\infty} 1/\Phi(n) < \infty$ *and set* $c_{\mathrm{K}} = \Lambda(\{0\}) \geq 0$ *and* $\lambda := \mu(\infty) \geq 0$.

(1) *If* $c_{\mathrm{K}} > 0$ *then* $\theta = 2\lambda/c_{\mathrm{K}}$,
$\rightarrow$ *same phase transition as for the 'fast' EFC.*
$\rightarrow$ *If* $\lambda = 0$ *then* $\theta = 0$ *(coming down from infinity=CDI).*

(2) *If* $\lambda > 0$ *and* $c_{\mathrm{K}} = 0$ *then* $\theta = \infty$ *(no CDI).*
$\rightarrow$ *Only the Kingman's part can face a fragmentation dislocating a block into infinitely many of its sub-blocks...*

Notice that

$$\Phi(n) \underset{n\to\infty}{\sim} \Psi(n) := \int_0^1 (e^{-nx} - 1 + nx)x^{-2}\Lambda(\mathrm{d}x).$$

Theorem (Λ & μ with regular variations)

If $\Phi(n) \underset{n\to\infty}{\sim} dn^{\beta+1}, \beta \in (0,1]$ and $\mu(n) \underset{n\to\infty}{\sim} \frac{b}{n^{\alpha+1}}$ with $\alpha \in (0,1)$ and $b > 0$, then

1. *if $\beta < 1-\alpha$, $\theta = \infty$*
2. *if $\beta > 1-\alpha$, $\theta = 0$*
3. *if $\beta = 1-\alpha$, $\theta = \frac{b}{d}\frac{1}{\alpha(1-\alpha)} \in (0,\infty)$:*
    1. *if $b/d > \alpha(1-\alpha)$, Π stays infinite,*
    2. *if $b/d < \alpha(1-\alpha)$, Π comes down from infinity.*

Explosion

Question (**Explosion**)

Assume $\#\Pi(0) < \infty$*:* $\exists t > 0; \#\Pi(t) = \infty$ *?*

Clearly if $\mu(\infty) > 0$, $\#\Pi$ explodes, but can we have 'continuous' explosion, that is to say, accumulation of large jumps on bounded intervals of time bringing the number of blocks to ∞ in finite time? Even when the coalescent part comes down from infinity?

The condition $\theta^\star < 1$ does not imply in general non-explosion of $\#\Pi$. Other taylor-made parameters have to be designed...

Explosion in pure fragmentation processes

In this case, $\#\Pi$ is a discrete branching process with splitting measure μ (no death). For any $n \geq 1$, set

$$\ell(n) := \sum_{k=1}^{n} \bar{\mu}(k).$$

Doney's condition for explosion of pure branching process (whose generator is $\mathcal{L}^f$) . If $\#\Pi(0) < \infty$ then

$$\exists t > 0 : \#\Pi(t) = \infty \Longleftrightarrow \sum_{n=1}^{\infty} \frac{1}{n\ell(n)} < \infty$$

We now introduce a technical condition on the function ℓ.
Condition $\mathbb{H}$: there exists an eventually non-decreasing positive function g such that:

$$\int^{\infty} \frac{\mathrm{d}x}{xg(x)} < \infty \text{ and } \ell(n) \geq g(\log n) \log n \text{ for large enough } n. \quad (\mathbb{H}).$$

Moreover

$$\mathbb{H} \Longrightarrow \text{Doney's condition.}$$

Explosion in simple EFC processes

Theorem (F. Zhou 2022: explosion and exit)

Assume condition $\mathbb{H}$ holds.

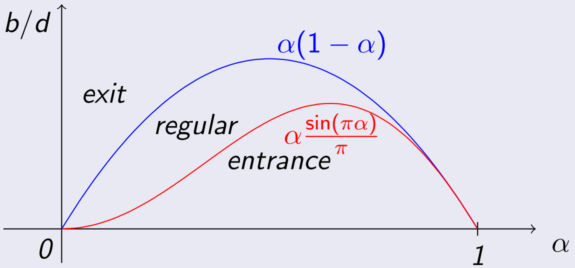
1. *If* $\rho := \limsup_{n\to\infty} \frac{\Phi(n)}{n\ell(n)} < 1/2$, *then* $(\#\Pi(t), t \geq 0)$ *explodes almost surely.*
2. *If furthermore,* $\sum_{n=2}^{\infty} \frac{1}{\Phi(n)} < \infty$ *and* $\rho < 1/4$, *then* ∞ *is an* **exit** *boundary (the process stays at* ∞ *after explosion).*

Theorem (F. Zhou 2022: non-explosion and entrance)

1. *If* $\sum_{n=2}^{\infty} \frac{n}{\Phi(n)} \bar{\mu}(n) < \infty$, *then* $(\#\Pi(t), t \geq 0)$ *does not explode almost surely.*
2. *If furthermore,* $\sum_{n=2}^{\infty} \frac{1}{\Phi(n)} < \infty$, *then* ∞ *is an* **entrance** *boundary (the process leaves* ∞ *and never returns there back).*

Theorem (F., Zhou 2022)

If $\Phi(n) \underset{n\to\infty}{\sim} dn^{2-\alpha}$ *and* $\mu(n) \underset{n\to\infty}{\sim} \frac{b}{n^{1+\alpha}}$*: then* ∞ *is:*



Example

Beta-coalescent versus stable branching.

In the regular case, the process Π leaves and returns to the set of partitions with infinitely many blocks (more precisely, it can only return to partitions with blocks of infinite size).

A condition for explosion in CTMCs

Let $(N(t), t \geq 0)$ be a Markov process valued in $\mathbb{N}$, with generator say $\mathscr{L}$.
For any $a > 0$ and any $n \in \mathbb{N}$, set

$$g_a(n) := n^{1-a} \text{ and } G_a(n) := -\frac{1}{n^{1-a}} \mathscr{L} g_a(n).$$

Theorem (F. Zhou 2022)

If there exist $a > 1$ and an eventually non-decreasing positive function g satisfying $\int^{\infty} \frac{\mathrm{d}x}{xg(x)} < \infty$ such that for all large enough n

$$G_a(n) \geq g(\log n) \log n,$$

then, setting

$$\tau_{\infty}^{+} = \inf\{t > 0 : N_{t-} = \infty\},$$

we have $\mathbb{P}_n(\tau_{\infty}^{+} < \infty) > 0$ for all large enough $n \in \mathbb{N}$.

A word on the proofs for the explosion

1. We start by establishing that big negative jumps due to the coalescence which would make decrease the number of blocks by a fixed proportion p cannot occur immediately;
2. We study the process ignoring those large negative jumps: its generator is

$$\mathscr{L}^p = \mathscr{L}^{c,p} + \mathscr{L}^f$$

with

$$\mathscr{L}^{c,p} f(n) = \sum_{k=2}^{[np]} \binom{n}{k} \lambda_{n,k} \left(f(n-k+1) - f(n) \right)$$

3. The function G_a associated to $\mathscr{L}^p$ is controlled by Φ and ℓ and by using the previous theorem and by optimizing in a and p, we get the explosion.

Conclusion and References

The last result for explosion is too short for studying the case with a Kingman component...

Thank you for your attention

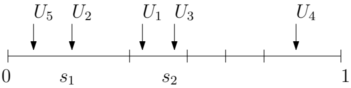
- J. Berestycki : Exchangeable fragmentation-coalescence processes and their equilibrium measure, EJP 2004
- Kyprianou, Pagett, Rogers, Schweinsberg: a phase transition in excursions from infinity in the fast fragmentation-coalescence process, AoP 2017
- Foucart : A phase transition in the coming down from infinity of simple EFCs, AAP 2022.
- Foucart and Zhou : On the explosion of the number of fragments in simple EFCs, AIHP 2022.
- Foucart and Zhou : On the boundary classification of Λ-Wright-Fisher processes with frequency-dependent selection (to appear in AHL 2023+)

Representation of μ_{Frag} and μ_{Coag}

<u>Paint-box:</u> Let $\mathcal{P}_{\mathrm{m}}^1 := \left\{(s_1, s_2, \ldots); s_1 \geq s_2 \geq \ldots \geq 0, \sum_{i=1}^{\infty} s_i = 1\right\}$.
Let $\mathrm{s} \in \mathcal{P}_{\mathrm{m}}^1$, $(U_i)_{i \geq 1}$ uniform iid on $(0,1)$: an s-paint-box is a random partition π:

$$i \sim j \text{ iff } U_i \text{ et } U_j \text{ fall in the same subinterval of } [0,1].$$

Denote by $\rho_{\mathrm{s}} :=$ the law of the s-paintbox π.



$\pi_{|[5]} = \{\{1,3\},\{2,5\},\{4\}\}$

The measures of coalescence and fragmentation take the form:

$$\mu_{\text{Coag}}(\mathrm{d}\pi) := c_{\mathrm{k}} \sum_{1 \leq i < j} \delta_{K(i,j)} + \int_{\mathcal{P}_{\mathrm{m}}} \rho_s(\cdot)\nu_{\text{Coag}}(\mathrm{d}s)$$

$$\mu_{\text{Frag}}(\mathrm{d}\pi) := c_{\mathrm{e}} \sum_{i \geq 1} \delta_{e(i)} + \int_{\mathcal{P}_{\mathrm{m}}} \rho_s(\cdot)\nu_{\text{Disl}}(\mathrm{d}s)$$

with $K(i,j) := (\cdots, \{i,j\}, \cdots)$ where $\cdots$ are singletons and $e(i) := (\{i\}, \mathbb{N} \setminus \{i\})$.

Number of blocks in simple EFCs

Coalescence. To each atom $(t,\pi^c)\in \mathrm{PPP}_C$, associate $(X_i, i\geq 1)$ s.t.

- $X_i=1$ if the block i takes part to the merging ($\{i\}$ is not a block of π^c)
- $X_i=0$ otherwise ($\{i\}$ is a block of π^c).

The X_i are exchangeable Bernoulli r.v's with de Finetti measure $x^{-2}\Lambda(\mathrm{d}x)$ where Λ is a finite measure on $[0,1]$. Given that $\#\Pi(t-)=n$,

$$\text{the jump : } n\mapsto n-k+1 \quad \text{has for rate} \quad \binom{n}{k}\lambda_{n,k}$$

with

$$\lambda_{n,k}:=\int_{]0,1]}x^k(1-x)^{n-k}x^{-2}\Lambda(\mathrm{d}x)+c_{\mathrm{K}}\mathbb{1}_{\{k=2\}}.$$

Indeed,

$$\#\mathrm{Coag}(\Pi(t-),\pi^c)=\#\{\cup_{i\in\pi^c_\ell\cap[n]}\Pi_i(t-),\ell\geq 1\}=n-\sum_{i=1}^{n}X_i+1$$

Fragmentation. Let $\mathrm{PPP}_F = \sum_{t>0} \delta_{(t,\pi^f,j)}$ be an independent PPP with intensity $\mathrm{d}t \otimes \mu_{\mathrm{Frag}}(\mathrm{d}\pi) \otimes \#(\mathrm{d}j)$. To each atom $(t,\pi^f,j) \in \mathrm{PPP}_F$, associate the r.v $k := \#\pi^f - 1$. This gives a PPP on $\mathbb{R}_+ \times \bar{\mathbb{N}} \times \mathbb{N}$ with intensity $dt \otimes \mu \otimes \#$, where

$$\mu := \text{image of } \mu_{\mathrm{Frag}} \text{ by the map } \pi \mapsto \#\pi - 1.$$

If $j \leq \#\Pi(t-)$ then the atom (t,π^f,j) is “seen” by $\Pi(t-)$ and by exchangeability: given that $\#\Pi(t-) = n$,

$$\text{the jump : } n \mapsto n+k \text{ has for rate } n\mu(k), \ \forall k \in \bar{\mathbb{N}}.$$

Indeed

$$\begin{aligned}\#\Pi(t) &= \#\mathrm{Frag}(\Pi(t-),\pi^f,j) \\ &= \#\Pi(t) - 1 + \#\{\Pi_j(t-) \cap \pi^f_i, 1 \leq i \leq \#\pi^f\}.\end{aligned}$$

For all $i \geq 1$: $\Pi_j(t-) \cap \pi^f_i \neq \emptyset$ a.s. Thus

$$\#\{\Pi_j(t-) \cap \pi^f_i, 1 \leq i \leq \#\pi^f\} - 1 = \#\pi^f - 1 = k$$