Epidemics on networks with preventive rewiring

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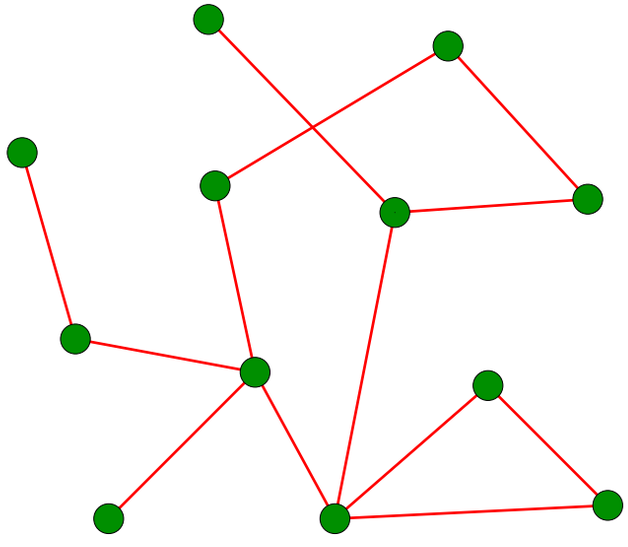
Network epidemic models



- Random graph of possible contacts
- Spread epidemic on graph. Interest often focussed on effect of properties of the random graph on disease dynamics.
- In this talk we analyse a model with adaptive dynamics, in which susceptibles may rewire edges away from infective neighbours.

Model with preventive rewiring

- Population of size n socially structured by an Erdős-Rényi graph $G(n, \frac{\mu}{n})$. Between each of the $\binom{n}{2}$ pairs of distinct nodes an edge is present independently with probability $\frac{\mu}{n}$, where $\mu > 1$ so giant component exists for large n.
- Markovian SIR (susceptible $\rightarrow$ infective $\rightarrow$ recovered) epidemic model with infection rate λ between neighbours and recovery rate γ.
- If a susceptible individual has an infective neighbour then that edge is rewired (to an individual chosen uniformly at random from the other $n - 2$ individuals in the population) at rate ω.
- Equivalently, an infective warns their neighbours independently at rate ω and warned susceptibles rewire such edges.
- Initially one infective and all other individuals susceptible.

(Jiang et al. (2019), cf. Britton et al. (2016), Leung et al. (2018))

Approximating branching process $\mathcal{B}$

- The process of infectives in the initial phase of an epidemic can be approximated by a branching process $\mathcal{B}$ in which
  - the lifetime of an individual $\sim \mathrm{Exp}(\gamma)$;
  - at birth an individual is assigned $\mathrm{Po}(\mu)$ infective edges;
  - an individual drops each infective edge independently at rate ω and infects down them independently at rate λ;
  - when an individual infects down an edge, a new individual is born and the edge is dropped.
- The basic reproduction number R_0 for the epidemic is given by the offspring mean of $\mathcal{B}$, viz.

$$R_0 = \mu \frac{\lambda}{\lambda + \omega + \gamma}.$$

Threshold theorem

Theorem 1 Let $T^{(n)}$ be the final size of the epidemic (i.e. the total number infected) and T be the total progeny of the branching process $\mathcal{B}$.

(a) $T^{(n)} \xrightarrow{p} T$ as $n \to \infty$.

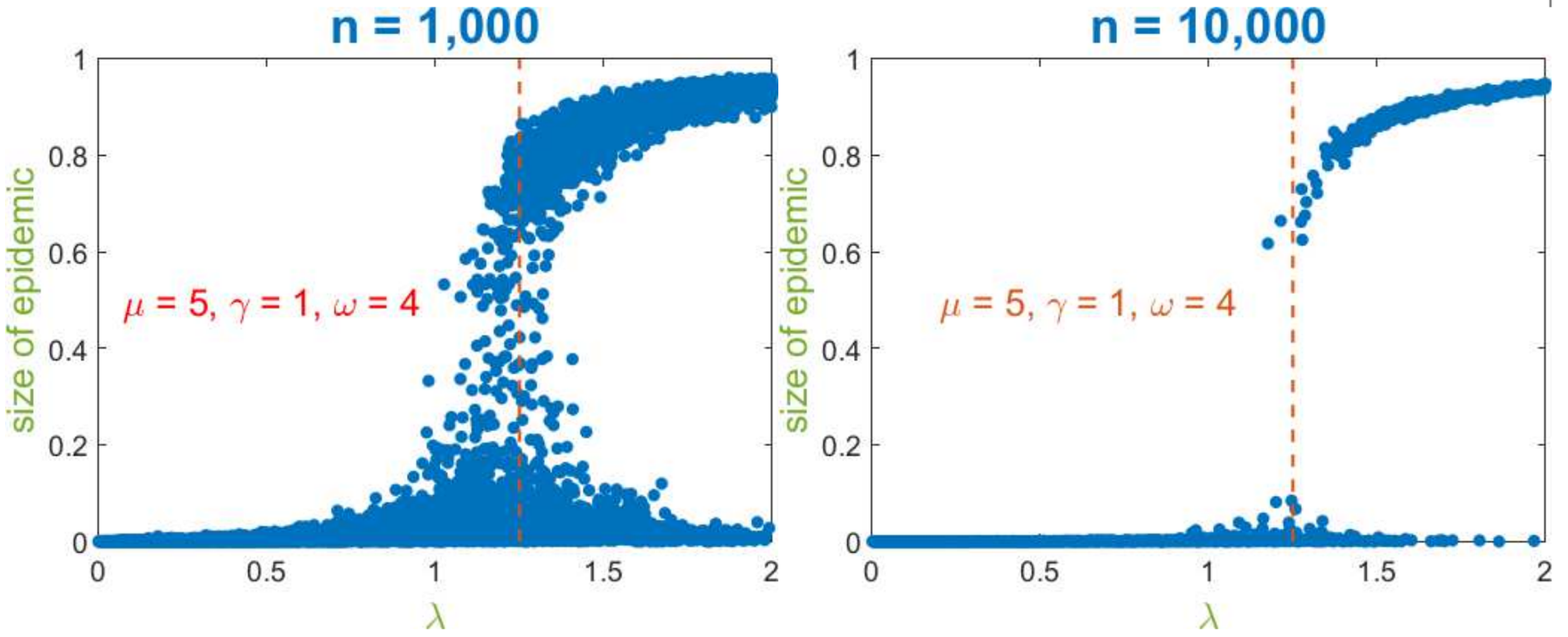
(b) $\lim_{n\to\infty} \mathrm{P}(T^{(n)} \geq \log n) = \mathrm{P}(T = \infty)$.

(c) Suppose $R_0 > 1$. Then there exists $\tau' = \tau'(\mu, \lambda, \gamma, \omega) > 0$ such that

$$\lim_{n\to\infty} \mathrm{P}(n^{-1}T^{(n)} \geq \tau' | T^{(n)} \geq \log n) = 1.$$

- We say that a major epidemic occurs if $T^{(n)} \geq \log n$.
- In the limit $n \to \infty$, a major epidemic occurs with non-zero probability if and only if $R_0 = \frac{\mu\lambda}{\lambda+\omega+\gamma} > 1$.
- If all other parameters are held fixed, $R_0 > 1 \iff \lambda > \lambda_C = \frac{\gamma+\omega}{\mu-1}$.

Final outcome of SIR model with rewiring



Final fraction infected in SIR model with rewiring on Erdős-Rényi graph with 5 initial infectives. Vertical line shows value of λ ($= \lambda_C$) so that $R_0 = 1$. Figure based on Jiang et al. (2019), Figure 4.

Construction of nearly-exact SIR model

Key ideas of construction

- Construct epidemic and partial network simultaneously.
- Only consider edges from infectives that are connected to susceptibles.
- Only keep track of the number of susceptible-susceptible rewired edges and not the individuals involved.

Construction of nearly-exact SIR model

- Let $S(t)$, $I(t)$ and $W(t)$ be the numbers of susceptibles, infectives and susceptible-susceptible rewired edges at time t and let $\mathcal{S}(t)$ be the set of susceptibles at time t.
- When an individual is infected it acquires $\mathrm{Po}(\mu_n S(t)/n)$ infectious edges. where $\mu_n = \mu\left(1 - \frac{\mu}{n}\right)^{-1}$.
- Infectious edges send warnings (and the infective loses the edge) at rate ω. When warning occurs, the edge is "rewired" to a susceptible, infective or recovered with probabilities $\frac{S(t)-1}{n-2}$, $\frac{I(t)-1}{n-2}$ and $\frac{n-S(t)-I(t)}{n-2}$. If the rewire is to
  - a susceptible then $W(t) \to W(t)+1$;
  - an infective then that infective gains an infectious edge;
  - a recovered then nothing further happens.

Construction of nearly-exact SIR model

- Each infectious edge transmits infection at rate λ. Then the edge is dropped, $(S(t), I(t)) \to (S(t)-1, I(t)+1)$ and
  - an individual (i_0 say), chosen uniformly at random from $\mathcal{S}(t-)$ is infected and $\mathcal{S}(t) \to \mathcal{S}(t) \setminus \{i_0\}$;
  - each other infectious edge is dropped independently with probability $\frac{1}{S(t-)}$;
  - individual i_0 acquires $R \sim \mathrm{Bin}\left(W(t-), \frac{2}{S(t-)}\right)$ further (rewired) infectious edges and $W(t) \to W(t) - R$.
- Infectives recover (and lose any remaining infectious edges) at rate γ.
- Construction is fully faithful to the original model if there is no multiple edge.
  - $\liminf_{n\to\infty} \mathrm{P}(\text{no multiple edge}) > 0$, so convergence in probability results transfer from construction to original model.

Construction of nearly-exact SIR model

- For $t \geq 0$ and $j = 0, 1, \ldots$, let $I_j^{(n)}(t)$ be the number of infectives with j infectious edges at time t.
- Then $X^{(n)} = \{(S^{(n)}(t), I_0^{(n)}(t), I_1^{(n)}(t), \ldots, W^{(n)}(t)) : t \geq 0\}$ is a density-dependent continuous-time Markov chain, with an infinite-dimensional state space, $E^{(n)}$ say, so the LLN in Ethier and Kurtz (1986) cannot be applied.
- Let $\boldsymbol{X}^{(n)}(t) = (S^{(n)}(t), I^{(n)}(t), I_E^{(n)}(t), W^{(n)}(t))$, where $I_E^{(n)}(t) = \sum_{j=0}^{\infty} j I_j^{(n)}(t)$ is the total number of infectious edges at time t.
- Can apply Darling and Norris (2008), Theorem 4.1, to obtain a WLLN for $\{\boldsymbol{X}^{(n)}(t) : t \geq 0\}$.

Weak law of large numbers

Theorem 2 Suppose $n^{-1}\boldsymbol{X}^{(n)}(0) \xrightarrow{p} \boldsymbol{x}(0)$ as $n \to \infty$, where $i(0) > 0$ and $i_E(0) > 0$. Then, for any $t_0 > 0$,

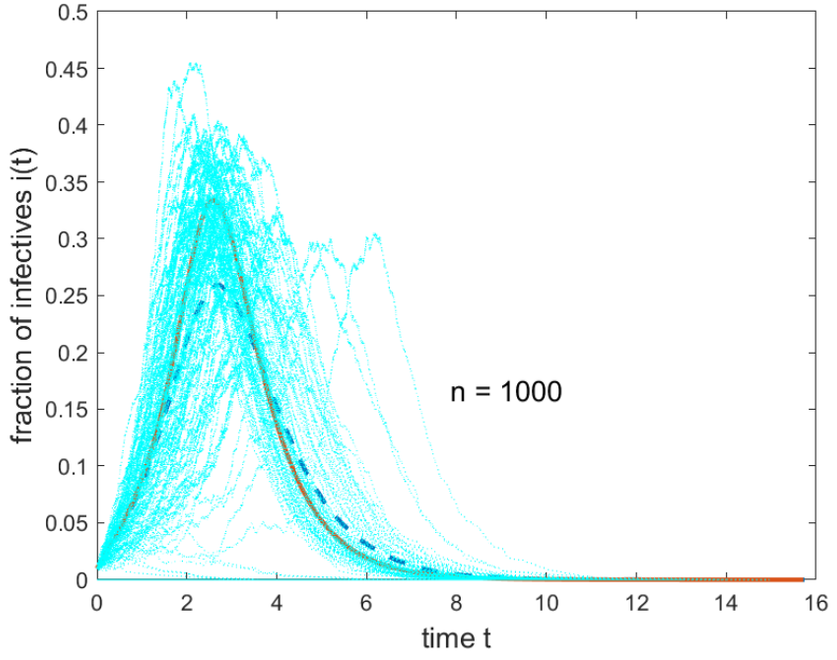
$$\sup_{0 \le t \le t_0} \left| n^{-1}\boldsymbol{X}^{(n)}(t) - \boldsymbol{x}(t) \right| \xrightarrow{p} 0 \quad \text{as } n \to \infty,$$

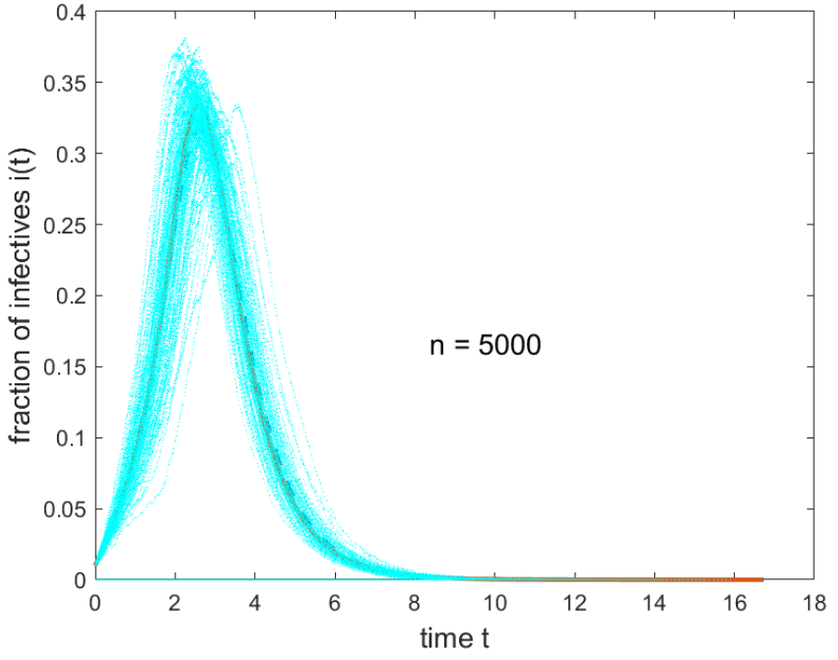
where $\boldsymbol{x}(t) = (s(t), i(t), i_E(t), w(t))$ is the solution of the ODE

$$\begin{aligned}
\frac{ds}{dt} &= -\lambda i_E, \\
\frac{di}{dt} &= \lambda i_E - \gamma i, \\
\frac{di_E}{dt} &= \lambda i_E \left[\mu s + 2\frac{w}{s} - 1 - \frac{i_E}{s} \right] - \gamma i_E - \omega i_E (1 - i), \\
\frac{dw}{dt} &= \omega i_E s - 2\lambda i_E \frac{w}{s},
\end{aligned}$$

having initial condition $\boldsymbol{x}(0) = (s(0), i(0), i_E(0), w(0))$.

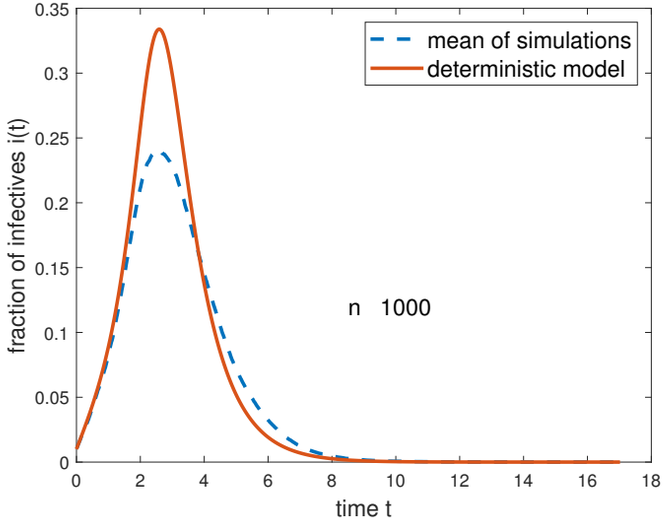
Illustration of WLLN

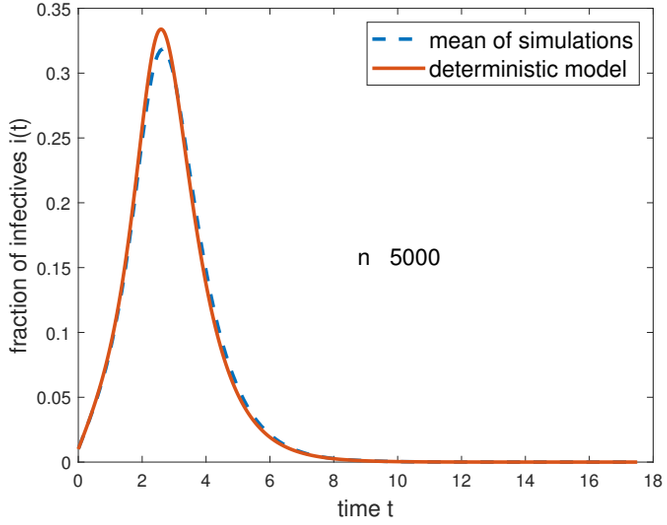


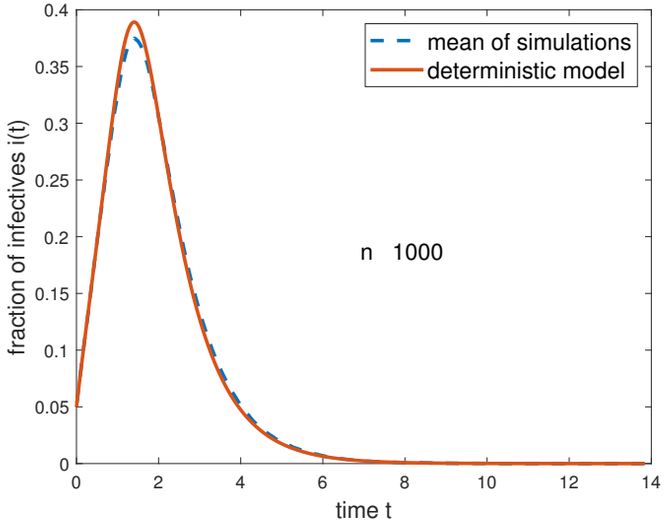


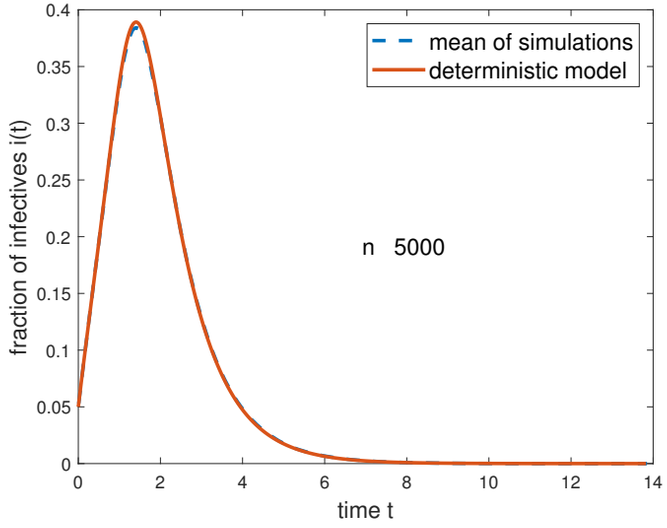
100 simulated realisations of trajectories of fraction infected in SIR model with $\mu = 5, \lambda = 1.5, \gamma = 1, \omega = 4$ ($R_0 = 1.1538$) and 1% initially infective. Also shown is the deterministic fraction $i(t)$ (solid curve) and the mean of the stochastic trajectories (dashed curve).

Illustration of WLLN









Upper row: 1% initially infective. Lower row: 5% initially infective

Final outcome of epidemic

- Let $\zeta^{(n)} = \inf\{t \geq 0 : I_E^{(n)}(t) = 0\}$, then the final size $T^{(n)}$ of the epidemic is given by $T^{(n)} = n - S^{(n)}(\zeta^{(n)})$.
- To study $T^{(n)}$ it is fruitful to consider the following random time-scale transformation of $X^{(n)}$ (cf. Watson (1980) and Janson et al. (2014)).
- Let $\xi = (n^S, n_0^I, n_1^I, \ldots, n^W)$ be a typical element of the state space $E^{(n)}$ of $X^{(n)}$ and $n^E = \sum_{k=0}^{\infty} k n_k^I$ and $\tilde{X}^{(n)}$ be the process with jump rates

$$\tilde{q}^{(n)}(\xi, \xi') = q^{(n)}(\xi, \xi')/(\lambda n^{-1} n^E) \qquad (\xi, \xi' \in E^{(n)}, \xi \neq \xi').$$

- The distribution of final size is invariant to this time transformation. We use $\tilde{\boldsymbol{X}}^{(n)} = \{(\tilde{X}^{(n)}(t), \tilde{I}^{(n)}(t), \tilde{I}_E^{(n)}(t), \tilde{W}^{(n)}(t)) : t \geq 0\}$ to analyse $T^{(n)}$.

Time-transformed deterministic approx

- The time-transformed deterministic approximation to $n^{-1}\tilde{\boldsymbol{X}}^{(n)}$ is

$$\begin{aligned}
\frac{d\tilde{s}}{dt} &= -1, \\
\frac{d\tilde{i}}{dt} &= 1 - \frac{\gamma}{\lambda}\frac{\tilde{i}}{\tilde{i}_E}, \\
\frac{d\tilde{i}_E}{dt} &= \mu\tilde{s} + 2\frac{\tilde{w}}{\tilde{s}} - 1 - \frac{\tilde{i}_E}{\tilde{s}} - \frac{\gamma}{\lambda} - \frac{\omega}{\lambda}(1-\tilde{i}), \\
\frac{d\tilde{w}}{dt} &= \frac{\omega\tilde{s}}{\lambda} - 2\frac{\tilde{w}}{\tilde{s}}.
\end{aligned}$$

- Final fraction infected $\tau = 1 - \tilde{s}(\tilde{\zeta})$, where $\tilde{\zeta} = \inf\{t > 0 : \tilde{i}_E(t) = 0\}$. (Note $\tilde{\zeta} < \infty$, unlike $\zeta = \inf\{t > 0 : i_E(t) = 0\}$.)
- Problems owing to this system not being Lipschitz in the neighbourhood of $\tilde{i}_E = 0$:
  - Darling and Norris (2008) Theorem 4.1 cannot be applied.
  - For epidemics with few initial infectives, τ depends on $\lim_{t\downarrow 0} \frac{\tilde{i}(t)}{\tilde{i}_E(t)}$.

Discontinuity at threshold $\lambda = \lambda_C$

- Consider modifications which bound the epidemic process with rewiring:
  - a lower bounding process, in which if a susceptible rewires an edge from one infective to another infective then the edge is dropped;
  - an upper bounding process, in which if a susceptible rewires an edge from an infective to a recovered individual then the edge to the infective is retained.
- Both modifications have the same approximating branching process $\mathcal{B}$, R_0 and λ_C as the original process, and yield time-transformed deterministic models for $(\tilde{s}(t), \tilde{i}_E(t), \tilde{w}(t))$ that are closed and Lipschitz.
- In a time transformed deterministic model, $\tilde{i}'_E(0) = 0 \iff \lambda = \lambda_C$. The final size is discontinuous (continuous) at $\lambda = \lambda_C$ if $\tilde{i}''_E(0) > 0$ (< 0) when $\lambda = \lambda_C$.

Discontinuity at threshold $\lambda = \lambda_C$

Theorem 3 Suppose that $R_0 > 1$.

(a) Suppose that $\omega > \gamma$ and $\mu > \frac{2\omega}{\omega - \gamma}$. Then there exists $\tau_0 = \tau_0(\mu, \gamma, \omega) > 0$ such that, conditional upon a major epidemic,

$$\lim_{n \to \infty} \mathrm{P}(n^{-1} T^{(n)} > \tau_0) = 1 \qquad \text{for all } \lambda > \lambda_C.$$

(b) Suppose that $2\omega \leq \gamma$ or $\mu \leq \frac{3\omega}{2\omega - \gamma}$. Then, for all $a > 0$, there exists $\lambda_1 > \lambda_C$ such that, conditional upon a major epidemic,

$$\lim_{n \to \infty} \mathrm{P}(n^{-1} T^{(n)} < a) = 1 \qquad \text{for all } \lambda \in (\lambda_C, \lambda_1).$$

Discontinuity at threshold $\lambda = \lambda_C$

Theorem 3 Suppose that $R_0 > 1$.

(a) Suppose that $\omega > \gamma$ and $\mu > \frac{2\omega}{\omega - \gamma}$. Then there exists $\tau_0 = \tau_0(\mu, \gamma, \omega) > 0$ such that, conditional upon a major epidemic,

$$\lim_{n\to\infty} \mathrm{P}(n^{-1}T^{(n)} > \tau_0) = 1 \qquad \text{for all } \lambda > \lambda_C.$$

(b) Suppose that $2\omega \leq \gamma$ or $\mu \leq \frac{3\omega}{2\omega - \gamma}$. Then, for all $a > 0$, there exists $\lambda_1 > \lambda_C$ such that, conditional upon a major epidemic,

$$\lim_{n\to\infty} \mathrm{P}(n^{-1}T^{(n)} < a) = 1 \qquad \text{for all } \lambda \in (\lambda_C, \lambda_1).$$

Theorem 3′ (Chen, Hou and Yao (2022)). Theorem 3(b) holds if $\omega \leq \gamma$ or $\mu \leq \frac{2\omega}{\omega - \gamma}$.

Final outcome of epidemic

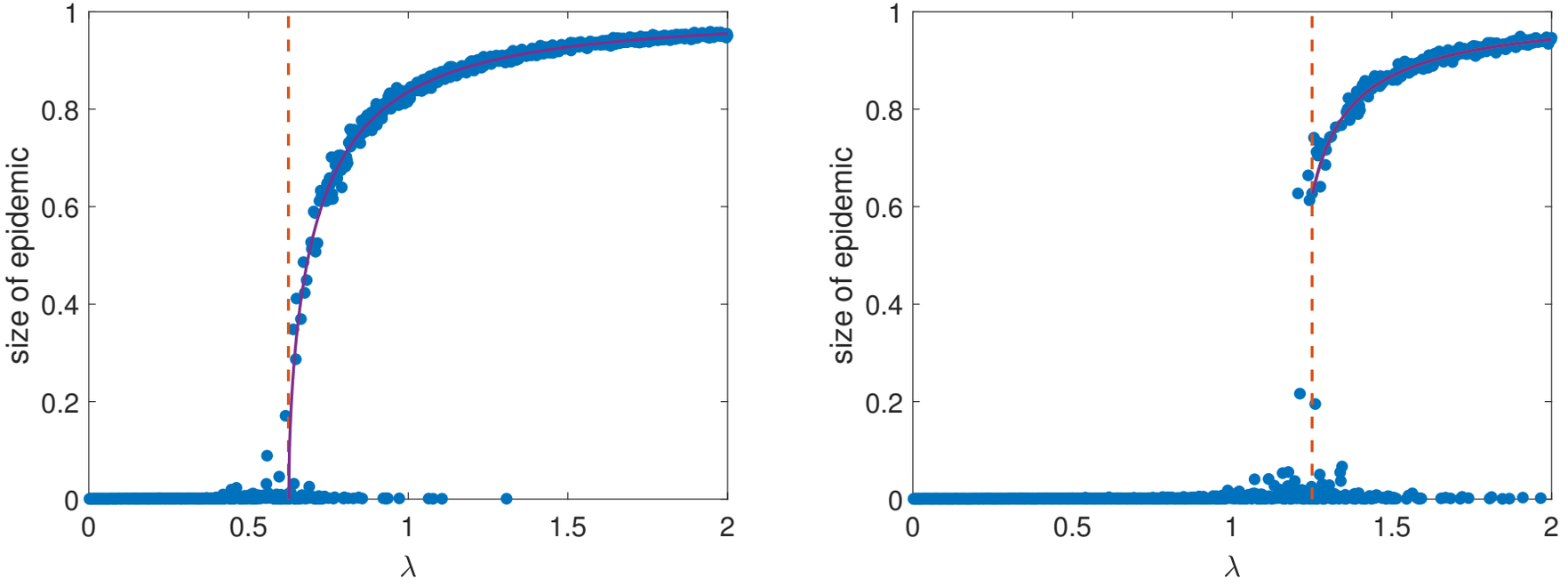
- Suppose that $R_0 > 1$. For $\varepsilon \in (0,1)$, let $x^\varepsilon(t) = (s^\varepsilon(t), i^\varepsilon(t), i_E^\varepsilon(t), w^\varepsilon(t))$ be the solution of the deterministic model with $x^\varepsilon(0) = (1-\varepsilon, \varepsilon, L^{-1}\varepsilon, 0)$, where $L = \frac{\lambda}{\lambda(\mu-1)-\omega}$, and $\tau = 1 - \lim_{\varepsilon \downarrow 0} s^\varepsilon(\infty)$.
- L is the almost sure limit of $I(t)/I_E(t)$ as $t \to \infty$ in the approximating branching process, conditional upon non-extinction.

Conjecture 1 Conditional upon a major epidemic,

$$n^{-1}T^{(n)} \xrightarrow{p} \tau \quad \text{as } n \to \infty.$$

Proved in Chen, Hou and Yao (2022) when $\omega \le \gamma$ or $\mu \le \frac{2\omega}{\omega-\gamma}$, i.e. when there is not a discontinuity at the threshold $\lambda = \lambda_C$.

Final outcome of SIR model with rewiring



1,000 simulations of final size of SIR epidemic when $n = 10,000, \mu = 5, \gamma = 1, \alpha = 1$ and varying λ; $\omega = \frac{3}{2}$ in the left panel and $\omega = 4$ in the right panel. Each simulation was started with 5 infectives. Solid curves show limiting fraction infected predicted by Conjecture 1.

SI model

- Suppose removal rate $\gamma = 0$ so infectives remain so forever, and $I^{(n)}(t) = n - S^{(n)}(t)$ and $i(t) = 1 - s(t)$ for all $t \geq 0$.
- Time-transformed ODE for $(\tilde{s}(t), \tilde{i}_E(t), \tilde{w}(t))$ is Lipschitz and admits a closed-form solution.

Theorem 4 (a) Suppose $R_0 > 1$. Then conditional upon a major epidemic,

$$n^{-1}T^{(n)} \xrightarrow{p} \tau \quad \text{as } n \to \infty,$$

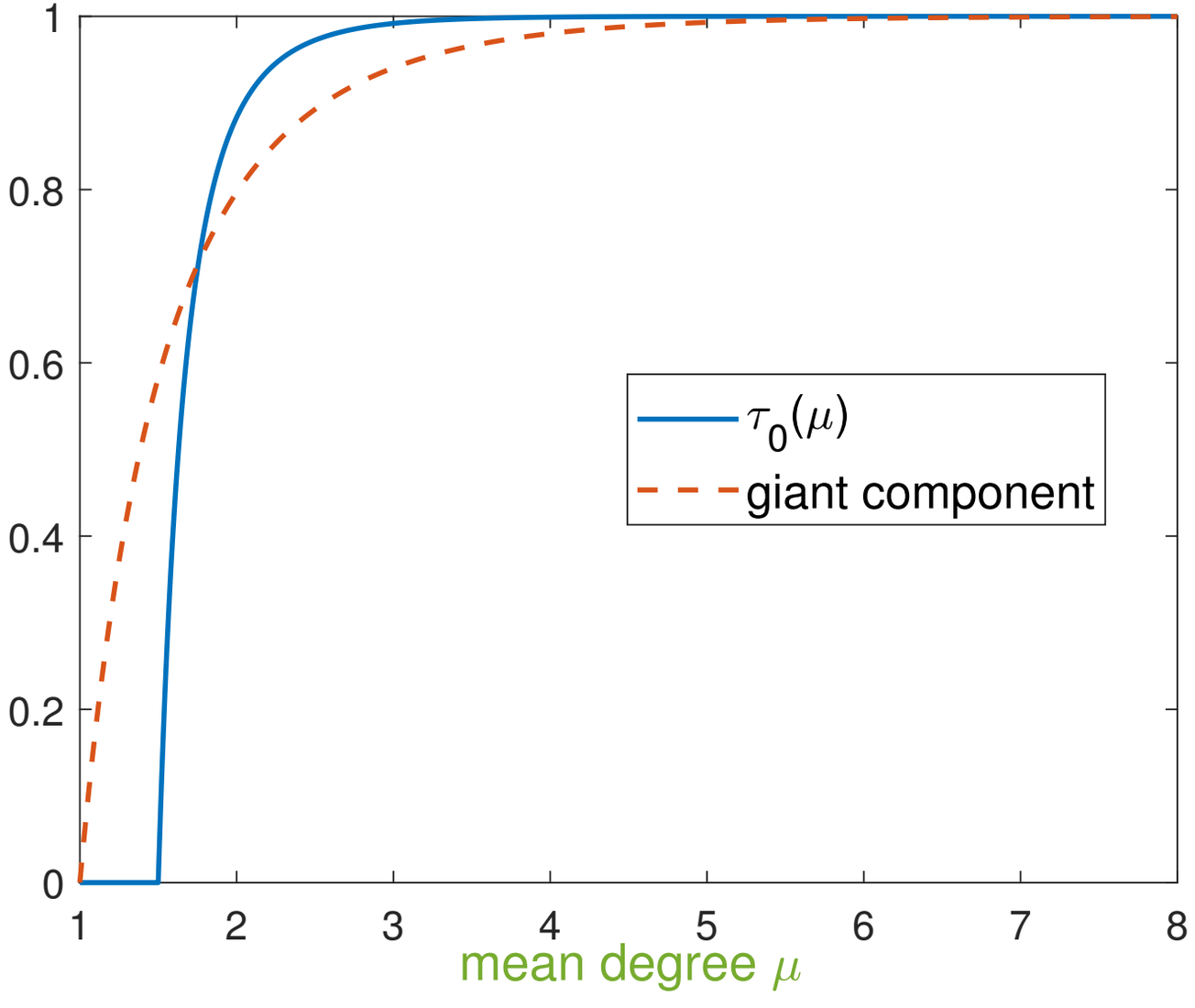
where $\tau = \tau_{\mathrm{SI}}(\mu, \lambda, \omega)$ is the unique solution in $(0, 1)$ of

$$1 - \tau = \exp\left(-\frac{\tau(\mu\lambda + \omega)}{\lambda + 2\omega(1 - \tau)}\right).$$

(b) Provided $\omega > 0$, $\tau_{\mathrm{SI}}(\mu, \lambda, \omega) \to \tau_0(\mu)$ as $\lambda \downarrow \lambda_C$ $(= \frac{\omega}{\mu - 1})$, where

$$\tau_0(\mu) > 0 \iff \mu > \frac{3}{2}.$$

$$\tau_0(\mu) = \lim_{\lambda \downarrow \lambda_C} \tau_{\mathrm{SI}}(\mu, \lambda, \omega)$$



Dependence of final size on ω

- Recall that $R_0 = \frac{\mu\lambda}{\lambda+\omega}$. Fix $\mu > 1, \lambda > 0$ and let $\omega_C = (\mu - 1)\lambda$. Then

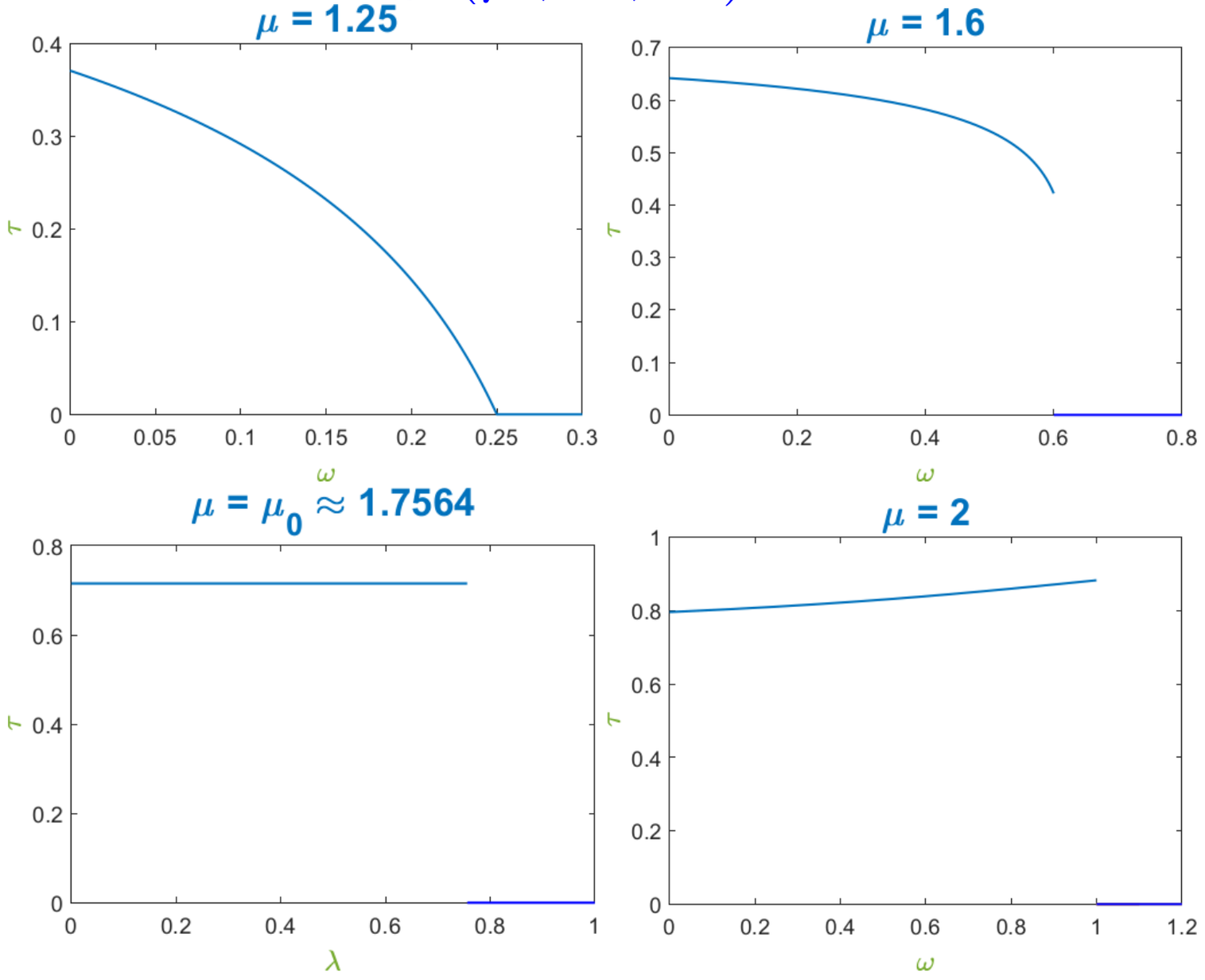
$$R_0 > 1 \iff \omega \in [0, \omega_C).$$

- Let μ_0 (≈ 1.7564) be the unique solution in $[1, \infty)$ of $2\mu = \mathrm{e}^{\mu - \frac{1}{2}}$. Then for $\omega \in [0, \omega_C)$,

$$\tau_{\mathrm{SI}}(\mu, \lambda, \omega) \begin{cases} \text{decreases with } \omega & \text{if } \mu < \mu_0 \quad \text{rewiring beneficial,} \\ \text{constant with } \omega & \text{if } \mu = \mu_0 \quad \text{rewiring neutral,} \\ \text{increases with } \omega & \text{if } \mu > \mu_0 \quad \text{rewiring harmful.} \end{cases}$$

- Note $\tau_{\mathrm{SI}}(\mu, \lambda, 0) =$ size of giant component of Erdős-Rényi graph $G(n, \frac{\mu}{n})$ for all $\lambda > 0$.

Final size $\tau_{\mathrm{SI}}(\mu, \lambda, \omega)$ when $\lambda = 1$



Four regimes are: (a) $1 < \mu \leq 1.5$; (b) $1.5 < \mu < \mu_0$; (c) $\mu = \mu_0$; and (d) $\mu > \mu_0$, where $\mu_0 > 1$ solves $2\mu = \mathrm{e}^{\mu - \frac{1}{2}}$.

Rewiring only to susceptibles

- Suppose that when a susceptible rewires an edge away from an infective, they rewire to an individual chosen uniformly at random from the other susceptibles.
- The deterministic approximation becomes

$$\begin{aligned}
\frac{ds}{dt} &= -\lambda i_E, \\
\frac{di}{dt} &= \lambda i_E - \gamma i, \\
\frac{di_E}{dt} &= \lambda i_E \left[\mu s + 2\frac{w}{s} - 1 - \frac{i_E}{s} \right] - \gamma i_E - \omega i_E, \\
\frac{dw}{dt} &= \omega i_E - 2\lambda i_E \frac{w}{s}.
\end{aligned}$$

- The equations for (s, i_E, w) form a closed system.

Rewiring only to susceptibles - final size

- The time transformed deterministic model for $(\tilde{s}(t), \tilde{i}_E(t), \tilde{w}(t))$ is Lipschitz. Its solution with initial condition $(\tilde{s}(0), \tilde{i}_E(0), \tilde{w}(0)) = (1, 0, 0)$ is

$$\tilde{s}(t) = 1 - t, \quad \tilde{i}_E(t) = \tilde{s}(t)\tilde{g}(\tilde{s}(t)), \quad \tilde{w}(t) = \frac{\omega\alpha}{\lambda}\tilde{s}(t)(1 - \tilde{s}(t)).$$

  where

$$\tilde{g}(s) = \left(1 + \frac{\gamma - \omega}{\lambda}\right) \log \tilde{s} + \left(\mu - \frac{2\alpha}{\lambda}\right)(1 - \tilde{s}).$$

- Note that $\tilde{i}_E(t) = 0 \iff \tilde{s}(t) = 0$ or $\tilde{g}(\tilde{s}(t)) = 0$.
- The equation $\tilde{g}(s) = 0$ has 0 or 1 solution in $(0, 1)$. If it has 0 solution then, in the model in real time, the final fraction susceptible $s(\infty) = 0$, otherwise it is given by the solution of $\tilde{g}(s) = 0$ in $(0, 1)$.

Rewiring only to susceptibles - final size

Theorem 5 Suppose that $R_0 = \frac{\mu\lambda}{\lambda+\omega+\gamma} > 1$. Then, conditional upon a major epidemic,

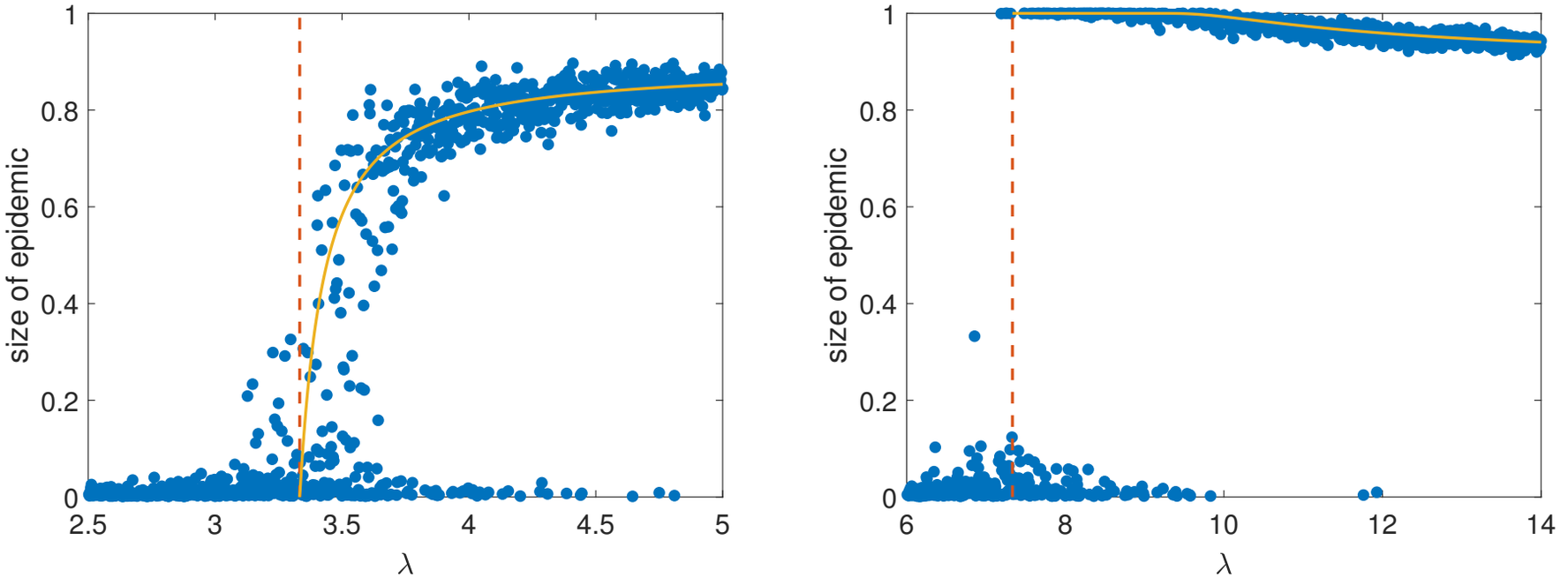
$$n^{-1}T^{(n)} \xrightarrow{p} \tilde{\tau} = \tilde{\tau}(\mu, \lambda, \gamma, \omega) \quad \text{as } n \to \infty,$$

where

(a) if $\mu(\gamma - \omega) + 2\omega \geq 0$ then, for all $\lambda > \lambda_C$, $\tilde{\tau}$ is given by the unique solution in $(0, 1)$ of $\tilde{g}(1 - x) = 0$, and is continuous at $\lambda = \lambda_C$;

(b) if $\mu(\gamma - \omega) + 2\omega < 0$ then $\tilde{\tau} = 1$, for $\lambda_C < \lambda \leq \omega - \gamma$, and $\tilde{\tau}$ is given by the unique solution in $(0, 1)$ of $\tilde{g}(1 - x) = 0$, for $\lambda > \omega - \gamma$.

Final outcome of SIR model with rewiring



1,000 simulations of final size of SIR epidemic with rewiring only to susceptibles when $n = 10,000, \mu = 2.5, \gamma = 1, \alpha = 1$ and varying λ; $\omega = 4$ in the left panel and $\omega = 10$ in the right panel. Each simulation was started with 10 infectives. Solid curves show limiting fraction infected predicted by Theorem 5.

Concluding comments

- All results generalise to the model in which warned susceptibles rewire the edge with probability $\alpha \in (0, 1)$ and drop it otherwise.
- Approximating deterministic model is equivalent to a pair-approximation model.
- Extension to other network models, e.g. configuration model (see Yao and Durrett (2022) for SI model).
- $R_0 < 1$ may not prevent a large epidemic unless n is very large.

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