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Power fractional laws in branching models

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Reminder: Linear fractional distributions

- The linear fractional distribution $\mathrm{LF}(a,b)$, with parameters $a,b>0,\ a+b\geq 1$, is a mixture of a point mass at 0 and a geometric distribution on the positive integers $\mathbb{N}$, denoted Geom_+. More precisely,

$$\mathrm{LF}(a,b)\ =\ \tfrac{a+b-1}{a+b}\,\delta_0\ +\ \tfrac{1}{a+b}\,\mathrm{Geom}_+\Big(\tfrac{a}{a+b}\Big).$$

- It has generating function (g.f.)

$$f(s)\ =\ \frac{a+(b-1)(1-s)}{a+b(1-s)}\ =\ 1-\left[\frac{a}{1-s}+b\right]^{-1},$$

mean $m=a^{-1}$ and variance $a^{-2}(2b+a-1)$.

- It is a pure geometric law iff $a+b=1$, namely

$$\mathrm{LF}(a,1-a)\ =\ \mathrm{Geom}_+(a).$$

A generalization: Power fractional distributions

- The power fractional distribution $\mathrm{PF}(\theta, a, b)$ has three parameters, viz.

$$\theta \in (0,1], \quad \text{and} \quad a, b > 0,\ a + b \geq 1,$$

- and g.f. f of the form

$$f(s) = 1 - \left[\frac{a}{(1-s)^{\theta}} + b\right]^{-1/\theta}, \quad s \in [0, \gamma).$$

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- Their goal: To give more general class of g.f.'s that are stable under iteration. (will return to this)
- The linear fractional distribution $\mathrm{LF}(a, b)$ appears as a special case: $\mathrm{LF}(a, b) = \mathrm{PF}(1, a, b)$.

A generalization: Power fractional distributions

- The first derivative of f equals

$$f'(s) \;=\; a\left[\frac{1}{a+b(1-s)^{\theta}}\right]^{(\theta+1)/\theta}, \quad s\in[0,\gamma),$$

- giving $f'(1)=a^{-1/\theta}$.
- for $\theta\in(0,1)$, all higher order derivatives at 1 are infinite!
- Confirmed by the following result about counting density and tails:

Power law tails for $0 < \theta < 1$

Power fractional distributions $\mathrm{PF}(\theta, a, b)$ with $0 < \theta < 1$ exhibit power law tail behavior (of order $1 + \theta$):

Let $(p_n)_{n \geq 0} = \mathrm{PF}(\theta, a, b)$ for $0 < \theta < 1$ and $a, b > 0$ such that $a + b \geq 1$. Then

$$(1) \qquad p_n \asymp n^{-(2+\theta)} \quad \text{as } n \to \infty.$$

If $a/(a + b) < \theta$, then $(n(n-1)p_n)_{n \geq 2}$ is decreasing and

$$(2) \qquad p_n \simeq cn^{-(2+\theta)} \quad \text{as } n \to \infty,$$

where $c = a^{-(\theta+1)/\theta} b(\theta + 1)/\Gamma(1 - \theta)$.

The Sibuya distribution

The Sibuya distribution $\mathrm{Sib}(a)$ for $a \in (0,1)$, named after Sibuya [4], has support $\mathbb{N}$, mean $m = \infty$, and g.f.

$$f(s) \;=\; 1 - (1-s)^a, \quad s \in [0,1].$$

It appears as a particular power-fractional law with $\theta = 0$ (not immediate, limiting case not discussed here).

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Mentioned here because ...

A Sibuya sum of iid power fractionals has the same law as a linear fractional sum of iid Sibuyas

A distributional relation

Here is the precise statement:

Fixing any $\theta \in (0, 1)$ and $a, b > 0$ with $a + b \geq 1$, the relation

$$\sum_{k=1}^{S} X_k \stackrel{d}{=} \sum_{k=1}^{Y} S_k$$

holds true for independent r.v.'s X, Y and $X_n, S_n, n = 1, 2, \ldots$, such that

- the law of $X, X_1, X_2, \ldots$ is $\mathrm{PF}(\theta, a, b)$ (with g.f. f),
- the law of Y is $\mathrm{LF}(a, b)$ (with g.f. g),
- and the law of $S, S_1, S_2, \ldots$ is $\mathrm{Sib}(\theta)$ (with g.f. h).

In terms of g.f.'s, the relation reads

$$h \circ f = g \circ h, \quad \text{or} \quad f = h^{-1} \circ g \circ h.$$

Therefore, $\mathrm{PF}(\theta, a, b)$ may be called a conjugation of $\mathrm{LF}(a, b)$ by means of a Sibuya law.

Stability under iteration

The g.f. f of PF(θ, a, b) satisfies the equation

$$\frac{1}{(1-f(s))^\theta} = \frac{a}{(1-s)^\theta} + b$$

and does indeed show stability under iteration. For $n = 2$, we find for $f^2(s) = f(f(s))$ that

$$\frac{1}{(1-f(f(s)))^\theta} = \frac{a}{(1-f(s))^\theta} + b = a\left(\frac{a}{(1-s)^\theta} + b\right) + b$$
$$= \frac{a^2}{(1-s)^\theta} + ab + b$$

and then for general $n \geq 2$

$$\frac{1}{(1-f^n(s))^\theta} = \frac{a^n}{(1-s)^\theta} + a^{n-1}b + a^{n-2}b + \ldots + ab + b$$

Stability under iteration

For two not necessarily identical power fractional g.f.'s $f \sim \mathrm{PF}(\theta, a, b)$ and $g \sim \mathrm{PF}(\theta, c, d)$, one finds accordingly

$$\frac{1}{(1 - f(g(s)))^\theta} = \frac{ac}{(1-s)^\theta} + ad + b$$

Observation: The parameter evolution does not depend on θ and is therefore the same as in the linear fractional case.

In terms of random variables, the above identity means that, given independent $Y \stackrel{d}{=} \mathrm{PF}(\theta, a, b)$ and $X_k \stackrel{d}{=} \mathrm{PF}(\theta, c, d)$ for $k \in \mathbb{N}$,

$$\sum_{k=1}^{Y} X_k \stackrel{d}{=} \mathrm{PF}(\theta, ac, ad + b)$$

GWP with power fractional offspring law

Consequence: if $(Z_n)_{n\geq 0}$ is a GWP with offspring law $\mathrm{PF}(\theta.a,b)$ and $Z_0 = 1$, then

$$Z_n \stackrel{d}{=} \mathrm{PF}\left(\theta, a^n, \sum_{k=1}^{n} a^{k-1} b\right) = \mathrm{PF}\left(\theta, a^n, \frac{b(a^n-1)}{a-1}\right)$$

in the noncritical case $a \neq 1$, and

$$Z_n \stackrel{d}{=} \mathrm{PF}\left(\theta, 1, bn\right)$$

in the critical case $a = 1$.

Extinction probability in the supercritical case $a < 1$:

$$q = 1 - \left(\frac{1-a}{b}\right)^{1/\theta}.$$

Supercritical GWP with power fractional offspring law

The normalized sequence and L^{1+}-bounded martingale

$$W_n := \frac{Z_n}{\mathbb{E}Z_n} = a^{n/\theta} Z_n, \quad n \geq 0$$

converges a.s. to a random variable W with $\mathbb{P}(W = 0) = q$ and Laplace transform

$$\varphi(u) = \mathbb{E}e^{-uW_\infty} = 1 - \left[\frac{1}{u^\theta} + \frac{b}{1-a}\right]^{-1/\theta}, \quad u \geq 0.$$

The associated distribution is called continuous power fractional law and abbreviated CPF. Notice that

$$\frac{1}{(1-\varphi(u))^\theta} = \frac{1}{u^\theta} + \frac{b}{1-a}.$$

This suggests to define $\mathrm{CPF}(\theta, \alpha, \beta)$ for $\theta \in (0, 1]$, $\alpha > 0$ and $\beta \geq 1$ as the distribution on $[0, \infty)$ with Laplace transform φ satisfying

$$\frac{1}{(1-\varphi(u))^\theta} = \frac{\alpha}{u^\theta} + \beta$$

or, equivalently,

$$\varphi(u) = (1-\beta^{-1/\theta}) + \beta^{-1/\theta} \underbrace{\left[1 - \left(1 - \frac{\alpha/\beta}{u^\theta + \alpha/\beta}\right)^{1/\theta}\right]}_{=:\mathrm{CPF}_+(\theta,\alpha/\beta)},$$

which in turn means that

$$\mathrm{CPF}(\theta, \alpha, \beta) = (1-\beta)^{-1/\theta}\delta_0 + \beta^{-1/\theta}\mathrm{CPF}_+(\theta, \alpha/\beta).$$

Continuous power fractional laws

The case $\theta = 1$ leads to continuous linear fractional laws:

$$\mathrm{CLF}(\alpha,\beta) := \mathrm{CPF}(1,\alpha,\beta) = (1-\beta)\delta_0 + \beta\, \mathit{Exp}(\alpha).$$

Finally, the result about the martingale limit W can now be restated as

$$W \stackrel{d}{=} \mathrm{CPF}(\theta, 1, b(1-a)^{-1}).$$

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Essentially unique and endogenous solution to the SFPE

$$Y \stackrel{d}{=} a^{1/\theta} \sum_{k=1}^{N} Y_k$$

with $N \stackrel{d}{=} \mathrm{PF}(\theta, a, b)$ independent of $Y_1, Y_2, \ldots$ which in turn are independent copies of Y.

Critical GWP with power fractional offspring law

In the critical case $a = 1$ with offspring law $\mathrm{PF}(\theta, 1, b)$, the following assertions are notable:

$$\lim_{n\to\infty} n^{1/\theta}\,\mathbb{P}(Z_n > 0) \;=\; b^{-1/\theta},$$
$$\lim_{n\to\infty} n^{-1/\theta}\,\mathbb{E}(Z_n|Z_n > 0) \;=\; b^{1/\theta},$$

and

$$\mathbb{P}\left(\frac{Z_n}{(bn)^{1/\theta}} \in \cdot \,\middle|\, Z_n > 0\right) \stackrel{w}{\to} \mathrm{CPF}_+(\theta, 1).$$

In the linear fractional case $\theta = 1$, the offspring variance equals $2b$.

Branching in varying PF environment

Explicit iteration is still possible when switching to branching in varying power fractional environment for fixed θ!

If $(Z_n)_{n\ge 0}$ is a GWPVE with offspring laws PF(θ, a_k, b_k), offspring g.f.'s f_k and $Z_0 = 1$, then Z_n has g.f. $f_1 \circ f_2 \circ \cdots \circ f_k$ and law

$$\mathrm{PF}\left(\theta, \prod_{k=1}^{n} a_k, \sum_{k=1}^{n}\left(\prod_{j=1}^{k-1} a_j\right) b_k\right)$$

Branching in varying PF environment

Interesting fact (not really new): Induced parameter evolution defines a deterministic walk on the affine linear group $\mathbb{R}_> \times \mathbb{R}$ with (non-commutative) multiplication

$$(a, b) \cdot (c, d) \;=\; (ac, ad + b).$$

Leads to random affine recursions and perpetuities when the environment becomes random, here

$$\text{i.i.d. } (A_1, B_1, (A_2, B_2) \ldots$$

Branching in PF random environment

- $(Z_n)_{n\geq 0}$ a GWP in i.i.d. power fractional random environment

  $$\mathbf{e} \;=\; (\mathbf{e}_n)_{n\geq 1}, \quad \text{where } \mathbf{e}_n = (A_n, B_n).$$

  Means that the quenched offspring law of individuals in generation $n-1$ is $\mathrm{PF}(\theta, A_n, B_n)$ with random g.f. f_n.
- $A, B > 0$ and $A + B \geq 1$ a.s.
- Putting $\mathbf{P} = \mathbb{P}(\cdot|\mathbf{e})$, we then have a.s.

  $$\mathcal{L}(Z_n|\mathbf{e}) \;=\; \mathbf{P}(Z_n \in \cdot) \;=\; \mathrm{PF}(\theta, \Pi_n, R_n),$$

  where

  $$(\Pi_n, R_n) := \left(\prod_{k=1}^{n} A_k, \sum_{k=1}^{n} \Pi_{k-1} B_k\right)$$

  for $n \in \mathbb{N}$.

A quick note on extinction

$$q_n(\mathbf{e}_{1:n}) \;:=\; \mathbb{P}(Z_n = 0|\mathbf{e}_{1:n}) \;=\; 1 - \frac{1}{(\Pi_n + R_n)^{1/\theta}}$$

$$q(\mathbf{e}) \;:=\; \lim_{n\to\infty} q_n(\mathbf{e}_{1:n}) \;=\; 1 - \frac{1}{\lim_{n\to\infty}(\Pi_n + R_n)^{1/\theta}}$$

Random affine recursions

Here are some basic facts about IFS generated by affine linear random functions $g_n(x) = A_n x + B_n$ with i.i.d. positive random coefficients A_n, B_n:

$$\text{Forward iterations}: g_{n:1}(x) = g_n \circ \ldots \circ g_1(x).$$
$$\text{Backward iterations}: g_{1:n}(x) = g_1 \circ \ldots \circ g_n(x).$$

They have identical marginals:

$$g_{n:1}(x) \stackrel{d}{=} g_{1:n}(x) \quad \text{for all } n \geq 1.$$

Forward iterations form a Markov chain which is asymptotically stable iff the (strictly increasing) backward iterations converge a.s. to a finite limit, which is given by

$$R_\infty := \sum_{k \geq 1} \Pi_{k-1} B_k$$

and called perpetuity.

Random affine recursions

Exact conditions for the a.s. convergence

$$R_n := \sum_{k=1}^{n} \Pi_{k-1} B_k \xrightarrow{n\to\infty} R_\infty$$

were given by Goldie & Maller [1]. Details not stated here, but essential condition (not surprising) is

$$\Pi_n \to 0 \quad \text{a.s.}$$

In the given branching context, where $A, B > 0$ and $A + B \geq 1$ must additionally hold, it easily follows that

$$R_\infty \geq 1 \quad \text{a.s.}$$

A duality result if $\Pi_n \to \infty$ a.s.

Defining $g_n^{(-1)}(x) := A_n^{-1}x + A_n^{-1}B_n$ for $n \in \mathbb{N}$ (which is not the inverse of g_n), the duality relation

$$\frac{R_n}{\Pi_n} = \frac{g_{1:n}(0)}{\Pi_n} = g_{n:1}^{(-1)}(0) \stackrel{d}{=} g_{1:n}^{(-1)}(0)$$
$$= \sum_{k=1}^{n} \Pi_k^{-1} B_k =: R_n^{(-1)}$$

holds for all $n \in \mathbb{N}$. Moreover,

$$\Pi_n \to \infty \text{ a.s.}$$

plus further conditions (again omitted) imply

$$R_\infty^{(-1)} := \sum_{k \geq 1} \Pi_k^{-1} B_k < \infty \quad \text{a.s.}$$

and

$$\frac{R_n}{\Pi_n} \stackrel{d}{\to} R_\infty^{(-1)}.$$

Back to branching in power fractional RE

Criticality classification embarks on the following trichotomy:

(C1) $R_\infty < \infty = R_\infty^{(-1)}$ a.s.

(C2) $R_\infty^{(-1)} < \infty = R_\infty$ a.s.

(C3) $R_\infty = R_\infty^{(-1)} = \infty$ a.s.

which in turn can be further characterized precisely in terms of A and B. We refrain from giving details, but based on this the following classification becomes reasonable:

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which in turn can be further characterized precisely in terms of A and B. We refrain from giving details, but based on this the following classification becomes reasonable:

$(Z_n)_{n\geq 0}$ is called

- supercritical under (C1);
- subcritical under (C2);
- critical/strongly critical under (C3), with a finer description omitted here.

Branching in power fractional RE

Note that the quenched logarithmic mean

$$\log \mathbf{E} Z_n \;=\; -\frac{1}{\theta}\log \Pi_n \;=\; -\frac{1}{\theta}\sum_{k=1}^{n}\log A_k \;=:\; \frac{1}{\theta}S_n, \quad n \geq 0$$

defines an ordinary random walk that can exhibit one of four fluctuation types, depending on the law of A. If $\mathbb{E}\log A$ exists, we have the following classification: $(Z_n)_{n\geq 0}$ is called

$$\begin{cases} \text{subcritical} & \text{if } \mathbb{E}\log A > 0, \\ \text{critical} & \text{if } \mathbb{E}\log A = 0 \text{ and } \mathbb{P}(A \neq 1) > 0, \\ \text{strongly critical} & \text{if } A = 1 \text{ a.s.}, \\ \text{supercritical} & \text{if } \mathbb{E}\log A < 0. \end{cases}$$

Back to branching in random environment

Assumptions and notation:

- $(Z_n)_{n\geq 0}$ a GWP in i.i.d. power fractional environment

$$\mathbf{e} \;=\; (\mathbf{e}_n)_{n\geq 1}, \quad \text{where } \mathbf{e}_n = (A_n, B_n).$$

  This means that the quenched offspring law of individuals in generation $n-1$ is $\mathrm{PF}(\theta, A_n, B_n)$ with random g.f. f_n.

- $A, B > 0$ and $A + B \geq 1$ a.s.
- We put $\mathbf{e}_{k:l} = (\mathbf{e}_k, \ldots, \mathbf{e}_l)$ for $k, l \geq 1$, $\mathbf{P} = \mathbb{P}(\cdot|\mathbf{e})$,

$$\mathbf{P}^{(1:n)} \;:=\; \mathbb{P}(\cdot|\mathbf{e}_{1:n}) \quad \text{and} \quad \mathbf{P}^{(n:1)} \;:=\; \mathbb{P}(\cdot|\mathbf{e}_{n:1})$$

  with corresponding expectations $\mathbf{E}$, $\mathbf{E}^{(1:n)}$ and $\mathbf{E}^{(n:1)}$. Then

$$\mathbf{P}^{(1:n)}((Z_0, \ldots, Z_n) \in \cdot) \;\stackrel{d}{=}\; \mathbf{P}^{(n:1)}((Z_0, \ldots, Z_n) \in \cdot)$$

  for each $n \in \mathbb{N}$.

Branching in power fractional RE

- Recalling $g_n(x) = g(\mathbf{e}_n, x) := A_n x + B_n,\ \Pi_n = \prod_{k=1}^n A_k$ and $R_n := \sum_{k=1}^n \Pi_{k-1} B_k$ for $n \in \mathbb{N}$, we have

$$\begin{aligned}\varphi \circ f_{1:n}(s) &= \frac{1}{(1 - f_{1:n}(s))^\theta} \\ &= \frac{\Pi_n}{(1-s)^\theta} + R_n = g_{1:n} \circ \varphi(s)\end{aligned}$$

  for $s \in [0,1)$, where $\varphi(x) = (1-x)^{-\theta}$.

- Shows that each $f_{1:n} = f_1 \circ \ldots \circ f_n$ is just a conjugation of the (backward) iteration $g_{1:n}$ of the random affine linear maps $A_n x + B_n$.
- The same statement holds for the forward iterations.
- Results are often (not always) nicer when stated in terms of backward iterations.

The subcritical case: quasi-stationary behavior

Let $(Z_n)_{n\ge 0}$ be subcritical, thus $R_\infty^{(-1)} < \infty = R_\infty$ and $\Pi_n \to \infty$ a.s. Put $h_n(s) = \mathbb{E}^{(1:n)}(s^{Z_n}|Z_n > 0)$ for $n \in \mathbb{N}$. Then

$$h_n(s) \;=\; \frac{f_{1:n}(s) - f_{1:n}(0)}{1 - f_{1:n}(0)} \;=\; 1 \;-\; \frac{1 - f_{1:n}(s)}{1 - f_{1:n}(0)}$$

and therefore

$$\begin{aligned}
\frac{1}{(1 - h_n(s))^\theta} &= \left[\frac{1 - f_{1:n}(0)}{1 - f_{1:n}(s)}\right]^\theta = \frac{\Pi_n(1-s)^{-\theta} + R_n}{\Pi_n + R_n} \\
&= \frac{\Pi_n}{\Pi_n + R_n} \cdot \frac{1}{(1-s)^\theta} + \frac{R_n}{\Pi_n + R_n} \\
&= \frac{1}{1 + R_n/\Pi_n} \cdot \frac{1}{(1-s)^\theta} + \frac{R_n/\Pi_n}{1 + R_n/\Pi_n}
\end{aligned}$$

for each $n \in \mathbb{N}$.

In other words, with probability one

$$\mathbf{P}^{(1:n)}(Z_n \in \cdot | Z_n > 0) \; = \; \mathsf{PF}\left(\theta, \frac{1}{1+R_n/\Pi_n}, \frac{R_n/\Pi_n}{1+R_n/\Pi_n}\right)$$

is power fractional on the positive integers $\mathbb{N}$.

However, it fluctuates in accordance with R_n/Π_n which in turn converges only in distribution. The same observation is made for the pertinent quenched survival probability:

$$\Pi_n^{1/\theta}\mathbf{P}^{(1:n)}(Z_n > 0) \; = \; \frac{1}{\mathbf{E}^{(1:n)}(Z_n | Z_n > 0)} \; = \; \frac{1}{(1+R_n/\Pi_n)^{1/\theta}}$$

almost surely.

This is an illustrative instance of where a reversal of the environment provides additional insight: Namely, this amounts to a replacement of R_n/Π_n by its a.s. convergent counterpart $R_n^{(-1)}$ (with the same law!). We have

$$\mathbf{P}^{(n:1)}(Z_n \in \cdot | Z_n > 0) \;=\; \mathsf{PF}\left(\theta, \frac{1}{1+R_n^{(-1)}}, \frac{R_n^{(-1)}}{1+R_n^{(-1)}}\right)$$

and accordingly

$$\Pi_n^{1/\theta}\mathbf{P}^{(n:1)}(Z_n > 0) \;=\; \frac{1}{\mathbf{E}^{(n:1)}(Z_n | Z_n > 0)} \;=\; \frac{1}{(1+R_n^{(-1)})^{1/\theta}}$$

almost surely.

The supercritical case: quenched limit behavior

Let $(Z_n)_{n\geq 0}$ be supercritical: $\Pi_n \to 0$, $R_\infty < \infty = R_\infty^{(-1)}$ a.s.

Then $W_n := \Pi_n^{1/\theta} Z_n$ for $n \geq 0$ forms nonnegative martingale with mean one under the quenched probability measure $\mathbf{P}$ (almost surely) and thus converges a.s. to a limit W.

The quenched law of W equals CPF$(\theta, 1, R_\infty)$, with Laplace transform

$$(3) \qquad \varphi(\mathbf{e}, u) = \mathbf{E}e^{-uW} = 1 - \left[\frac{1}{u^\theta} + R_\infty\right]^{-1/\theta}, \quad u \geq 0$$

and particularly also mean one.

Applications

- Multi-type branching
- Host-parasite coevolution (e.g. basic model studied by Kimmel and Bansaye), Kleine Büning.
- Stochastic Ricker model and quasi-stationarity (Högnäs)
- Two-sex branching models

Hose-parasite coevolution (optional)

Parasite evolution in a random cell line if parasites multiply in accordance with a linear fractional law (binary cell division):

- parasites multiply independently.
- parasite offspring law is $\mathrm{LF}(a, b)$.
- offspring of a parasite in a cell is randomly shared into the left or right daughter cell with probability p and $1 - p$, respectively.

Let $Z_n(v)$ denote the number of parasites sitting in cell $v = v_1 \cdots v_n \in \{0, 1\}^n$ for $n \in \mathbb{N}$ and $Z_0 = 1$. Then

... the law of $Z_n(v_1 \cdots v_n)$

$$\mathrm{LF}\left(\frac{a^n}{p^{s_n}(1-p)^{n-s_n}}, b\sum_{k=0}^{n-1}\frac{a^n}{p^{s_k}(1-p)^{k-s_k}}\right),$$

where $s_n = s_n(v) := \sum_{i=1}^n (1-v_i)$ for each $v \in \{0,1\}^n$ and n.

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random cell line $\to$ simple random walk $(S_n)_{n\geq 0}$.

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random cell line $\to$ simple random walk $(S_n)_{n\geq 0}$.

can be extended to random environment acting on offspring law and/or sharing mechanism.

Stochastic Ricker model (optional)

Ricker function: $R(x) = xe^{\alpha(1-x/K)}$, $\alpha, K > 0$. (Ricker 1954)
α = intrinsic growth rate, K = carrying capacity

$$\mathcal{L}(Z_n|Z_{n-1}) = \left(1 - \frac{Z_{n-1}e^{\alpha(1-Z_{n-1}/K)}}{1+r}\right)\delta_0 + \frac{Z_{n-1}e^{\alpha(1-Z_{n-1}/K)}}{1+r}\,\mathrm{Geom}_+\left(\frac{1}{1+r}\right)$$

Implies $\mathbb{E}(Z_n|Z_{n-1}) = Z_{n-1}e^{\alpha(1-Z_{n-1}/K)}$

$$\begin{aligned} f_n(s) &= \mathbb{E}\left(1 - \frac{Z_{n-1}e^{\alpha(1-Z_{n-1}/K)}}{1+r}\right) + \mathbb{E}\left(\frac{Z_{n-1}e^{\alpha(1-Z_{n-1}/K)}}{1+r}\right) g(s) \\ &= \left(1 - \frac{e^{\alpha(K-1/K)} f'_{n-1}(e^{-\alpha/K})}{1+r}\right) + \frac{e^{\alpha(K-1/K)} f'_{n-1}(e^{-\alpha/K})}{1+r} g(s) \end{aligned}$$

Stochastic Ricker model (optional)

This entails

$$Z_n \stackrel{d}{=} \text{LF}\left(\frac{e^{\alpha(K-1/K)} f'_{n-1}(e^{-\alpha/K})}{1+r}, \frac{r e^{\alpha(K-1/K)} f'_{n-1}(e^{-\alpha/K})}{1+r}\right)$$

for each $n \geq 1$. Moreover,

$$\mathbb{P}(Z_n \in \cdot | Z_n > 0) = \text{Geom}_+\left(\frac{1}{1+r}\right).$$

Thus, the quasi-stationary law of the sequence is positive geometric with parameter $1/(1+r)$.

Stochastic Ricker model (optional)

Can impose a random environment by making parameters α, r and K randomly change over time.


Thank You

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