LDP for the maximum of a two-speed BBM

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Collaborations with
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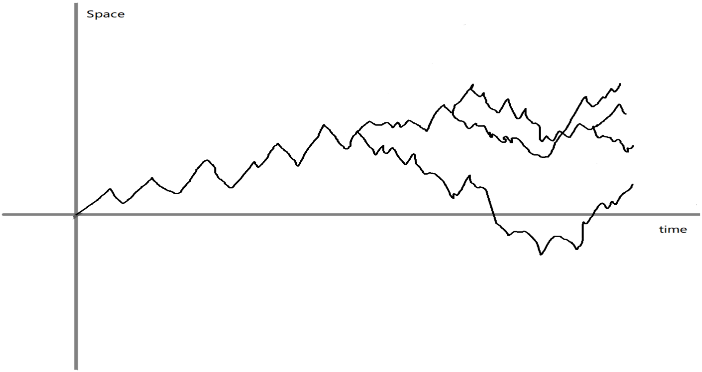
Contents

- Branching Brownian motion
- Two speed branching Brownian motion

Branching Brownian motion (BBM)

- Each particle moves as a BM.
- life time $\sim Exp.(1)$ with two offsprings.
- Each offspring performs the same behaviors (independently).
- M_t: maximal position of BBM at time t.



Pic. from M. Roberts

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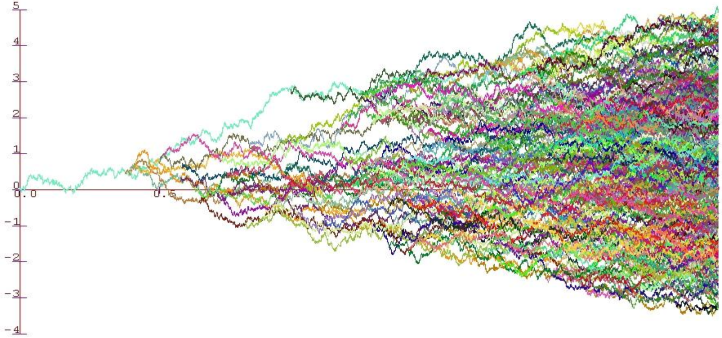

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- Berestycki, Brunet, Cortines and Mallein (2022): $\alpha \mapsto \text{Cst}(\alpha)$.
- Derrida, Meerson and Sasorov (2016): conjecture on $\text{Cst}'(1+)$.

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&= P(\text{there exists one particle which is higher than } \sqrt{2}\alpha t) \\
&\leq E[\mathcal{N}(t)]P(B_t \geq \sqrt{2}\alpha t) \\
&= e^t \frac{1}{\sqrt{2\pi t}} \int_{\sqrt{2}\alpha t}^{\infty} e^{-\frac{x^2}{2t}} dx \sim e^t \frac{1}{\sqrt{2\pi t}} e^{-\frac{2\alpha^2 t^2}{2t}} = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{t}} e^{-(\alpha^2-1)t}.
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- Lower bound: refined Bramson's argument on the F-KPP equation.
- Our goal: from BBM to two speed BBM

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Two speed branching Brownian motion

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- (regular) BM: $B_s \stackrel{d}{=} N(0,s),\ 0 \le s \le t.$

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where

$$\sigma(s) = \begin{cases} \sigma_1, & 0 < s < bt; \\ \sigma_2, & bt \le s \le t, \end{cases} \quad 0 < b < 1.$$

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- $$W_s \overset{d}{=} \begin{cases} N\left(0, \sigma_1^2 s\right), & 0 < s < bt; \\ N\left(0, \sigma_1^2 bt + \sigma_2^2(t-s)\right), & bt \le s \le t. \end{cases}$$

Weak convergence of the centered maximum

- Two speed BBM: Derrida and Spohn (1988), Fang and Zeitouni (2012), Bovier and Hartung (2014).

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$$M_t - m_t \quad \text{converges weakly},$$

$$m_t = \begin{cases} \sqrt{2}t - \frac{1}{2\sqrt{2}} \log t, & \text{if } \sigma_1 < \sigma_2, \\ \sqrt{2}(\sigma_1 b + \sigma_2(1-b))t - \frac{3}{2\sqrt{2}}(\sigma_1 + \sigma_2) \log t, & \text{if } \sigma_1 > \sigma_2. \end{cases}$$

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- Recall: for regular σ^2-BBM, $m_t = \sqrt{2}\sigma t - \frac{3\sigma}{2\sqrt{2}} \log t$.

Large deviation I: Chen, X., Chen, Z., H., Hartung (2023+)

- Assume $\sigma_1^2 b + \sigma_2^2(1-b) = 1$ and $\sigma_1 < \sigma_2$. Then, for any $\alpha > 1$,

$$\lim_{t\to+\infty} \sqrt{t} e^{(\alpha^2-1)t} P(M_t \geq \sqrt{2}\alpha t) = \text{ some constant.}$$

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- Parallel to regular BBM.
- Proof: Trivial.

Large deviation II: Chen, X., Chen, Z., H., Hartung (2023+)

Assume $\sigma_1^2 b + \sigma_2^2(1-b) = 1$ and $\sigma_1 > \sigma_2$. For any $\alpha > 1$, set $\alpha_1 := \alpha(b\sigma_1 + (1-b)\sigma_2)$.

- if $\alpha_1\sigma_2 > 1$, then

$$\lim_{t\to+\infty} \sqrt{t} e^{(\alpha_1^2-1)t} P(M_t \geq \sqrt{2}\alpha_1 t) = C_1;$$

- if $\alpha_1\sigma_2 = 1$, then

$$\lim_{t\to+\infty} t e^{(\alpha_1^2-1)t} P(M_t \geq \sqrt{2}\alpha_1 t) = C_2;$$

- if $\alpha_1\sigma_2 < 1$, then

$$\lim_{t\to+\infty} t^{\frac{1}{2}+\frac{3\sigma_2(\alpha_1-\sigma_2(1-b))}{2\sigma_1^2 b}} e^{\left(\frac{(\alpha_1-(1-b)\sigma_2)^2}{\sigma_1^2 b}-b\right)t} P(M_t \geq \sqrt{2}\alpha_1 t) = C_3.$$

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where $a_t = \sqrt{2}(\alpha_1 - \sigma_2(1-b))t + \frac{3\sigma_2}{2\sqrt{2}}\log(1-b)t$.
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Thanks