The survival probability of a weakly subcritical branching process in random environment

joint work with D. C. Pham

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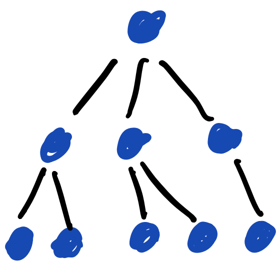
1. Introduction

A. One type branching process in fixed environment



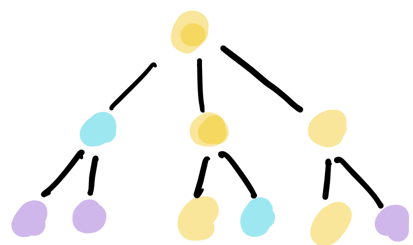
$(\xi_{n,k})_{n,k}$ iid $\qquad \mathbb{E}(\xi_{n,k}) = m, \; f_\xi = f$

$$Z_{n+1} = \begin{cases} 0 & \text{if } Z_n = 0 \\ \sum_{1}^{Z_n} \xi_{n+1,i} & \text{if } Z_n \neq 0 \end{cases} \qquad f_{Z_n} = f^{\circ n}$$

$$\mathbb{P}(Z_n = 0) = \underbrace{f \circ \cdots \circ f}_{n}(0) \to e \qquad q_n = \mathbb{P}(Z_n > 0)$$

Subcritical	Critical	Supercritical
$m < 1$	$m = 1$	$m > 1$
$e = 1$	$e = 1$	$e < 1$
$q_n \sim c\, m^n$	$q_n \sim \dfrac{2}{\sigma^2 n}$	

B. Multitype BP in fixed environment



$\xi_{n,k} = (\xi_{n,k}(i,j)),\ i,j \in \{1,\dots,p\}$

$M = (M(i,j)) \quad f_\xi = f = (f^1, \dots, f^p)$

$Z_n = (Z_n(1), \dots, Z_n(p)) \qquad f_{Z_n} = f \circ \dots \circ f$

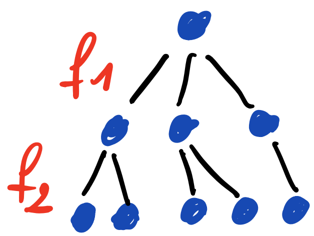
$e_n^i = \mathbb{P}(|Z_n| = 0 / Z_0 = e_i) = f^i \circ \dots \circ f(0)$

$q_n^i = \mathbb{P}(|Z_n| > 0 / Z_0 = e_i)$

$\rho = \rho(M) =$ spectral radius of M $= \lim_{+\infty} \|M^n\|^{1/n}$

Subcritical	Critical	Supercritical
$\rho(M) < 1$	$\rho(M) = 1$	$\rho(M) > 1$
$e^i = 1$	$e^i = 1$	$e^i < 1$
$q_n^i \sim c\,\rho^n$	$q_n^i \sim c/n$	

C. One type BP in random environment



$f_n = f_{\xi_{n,k}}$ $(f_n)_n$ iid

Annealed approach: $e_n = \mathbb{E}(f_1 \circ \dots \circ f_n(0)) = \mathbb{P}(Z_n = 0)$

$m_n = \mathbb{E}(\xi_{n,k} / f_n)$

Subcritical	**Critical**	**Supercritical**
$\mathbb{E}(\log m_n) < 0$	$\mathbb{E}(\log m_n) = 0$	$\mathbb{E}(\log m_n) > 0$
$e = 1$	$e = 1$	$e < 1$
3 subcases	$q_n \sim c/\sqrt{n}$	
Strong Intermediate Weak		

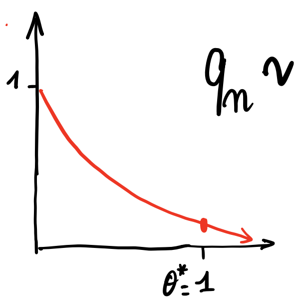
Subcritical case : the 3 subcases

Assume $\mathbb{E}(m_1) < +\infty$ $\quad E(\log m_1) < 0$.

$L : \theta \mapsto \mathbb{E}(m_1^{\theta})$ convex, $\quad L(0) = 1$, $L'(0) = \mathbb{E}(\log m_1)$

$\theta^* \in [0,1]$ s.t. $L(\theta^*) = \inf_{[0,1]} L(\theta) < 1$

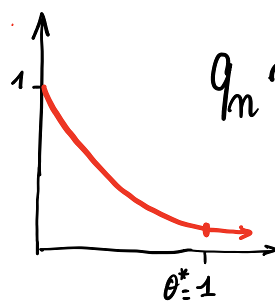
$L'(1) < 0 \quad \theta^* = 1$



$q_n \sim c\, L(\theta^*)^n$

strongly subcritical

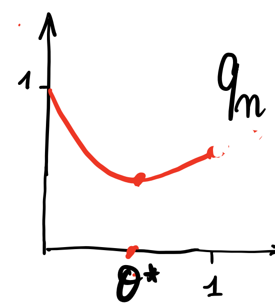
$L'(1) = 0 \quad \theta^* = 1$



$q_n \sim c\, L(\theta^*)^n / \sqrt{n}$

Intermediately subcritical

$L'(1) > 0 \quad \theta^* \in]0,1[$

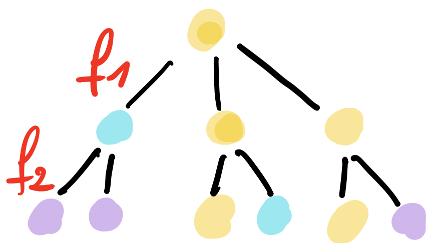


$q_n \sim c\, L(\theta^*)^n / n^{3/2}$

Weakly subcritical

Guivarch - Liu '90 $\qquad$ Geiger, Kersting, Vatutin 2003

D. Multitype BP in random environment



$f_n = (f_n^1, \dots, f_n^p)$ $(f_n)_n$ iid

$e_n^i = \mathbb{E}(f_1^i \circ \dots \circ f_n(0)) = \mathbb{P}(|Z_n|=0 / e_i)$

$M_n = (M_n(i,j))$ random matrix

$M_1 \dots M_n(i,j) = \mathbb{E}(Z_n(j) / Z_0 = e_i, f_1, \dots, f_n)$

$e_n^i = \mathbb{P}(|Z_n| = 0 / Z_0 = e_i)$

$\gamma = \lim_{+\infty} \frac{1}{n} \mathbb{E}(\log \|M_1 \dots M_n\|)$

Subcritical	Critical	Supercritical
$\gamma < 0$	$\gamma = 0$	$\gamma > 0$
$e = 1$	$e = 1$	$e < 1$
3 cases	$q_n^i \sim c_i / \sqrt{n}$	
Strong Intermediate Weak		

Subcritical case for multitype BP

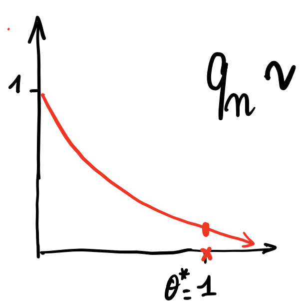
$\underline{\mathbb{E}(|M_1|) < +\infty}$

$L(\theta)$ replaced by $\lambda(\theta) = \lim_{+\infty} \mathbb{E}(|M_1 \cdots M_n|^{\theta})^{1/n}$

$\lambda(\theta)$ is log convex on $[0,1]$ $\quad \Lambda(\lambda) = \log \lambda(\theta)$

$\theta^* \in [0,1]$ s.t. $\lambda(\theta^*) = \inf \lambda(\theta)$

$\Lambda'(1) < 0 \quad \theta^* = 1$

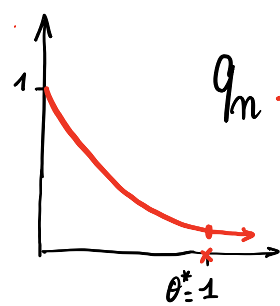


$q_n \sim c.\lambda(1)^n$

strongly subcritical

Vatutin - Wachtel '19

$\Lambda'(1) = 0 \quad \theta^* = 1$

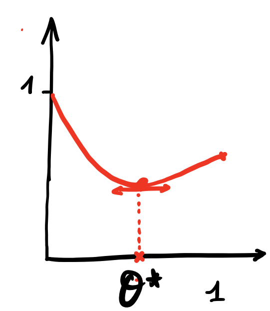


$q_n \asymp \lambda(1)^n / \sqrt{n}$

Intermediately subcritical

Dyakonova - Vatutin '20

$\Lambda'(1) > 0, \quad \theta^* \in]0,1[$



q_n ?

Weakly subcritical

II Statement

$(f_n)_n$ iid $\quad f_n =$ generating function of $\xi_{n,k} = (\xi_{n,k}(i,j))_{ij}$

$M_n = p \times p$ mean matrix of $\xi_{n,k}$

H1 - $\mathbb{E}(|M_1|) < +\infty$

H2 - Irreducibility

There exists no affine subspace $\mathcal{A}$ of $\mathbb{R}^p$ s.t $\mathcal{A} \cap (\mathbb{R}^+)^p$ is non empty, bounded and invariant under the action of elements of the support of μ

H3 - $\exists\, \Delta > 0$ s.t $\mathbb{P}$-as $\quad \frac{1}{\Delta} M_1(k,\ell) \leq M_1(i,j) \leq \Delta\, M_1(k,\ell)$

H4 - Weak subcriticality: $\quad \gamma_\mu < 0$ and $\Lambda'(1) > 0$

H5 - Non lattice assumption

H6 - Moment condition and others on $\xi_{n,k}$

Theorem Assume H1 - H6.

Then $\quad \mathbb{P}(|Z_n| > 0 / Z_0 = e_i) \asymp \dfrac{\lambda(\theta^*)^n}{n^{3/2}}$

III Sketch of the proof

A. Exponential change of measures

$X = \{ x = (x_1, \dots, x_p) \mid x_i \geq 0,\ x_1 + \dots + x_p = 1 \}$ $\qquad M_{1n} = M_1 \cdots M_n$

$P_\theta : \quad C(X) \to C(X)$

$\qquad \varphi \longmapsto P_\theta \varphi(x) = \mathbb{E}\left(|x M_1|^\theta \varphi(x \cdot M_1) \right)$

P_θ quasi-compact on the space of Hölder continuous functions on X with spectral radius $\lambda(\theta)$.

- $\nu_\theta P_\theta = \lambda(\theta) \nu_\theta \qquad P_\theta r_\theta = \lambda(\theta) r_\theta \qquad \nu_\theta$ probability measure $\qquad r_\theta > 0$ on X.
- $\overline{P}_\theta \varphi(x) = \dfrac{1}{\lambda(\theta)} \dfrac{1}{r(x)} \mathbb{E}\left(|x M_1|^\theta r_\theta(x \cdot M_1) \varphi(x \cdot M_1) \right)$

  $\overline{P}_\theta$ Markov operator $\rightsquigarrow (X_n^\theta)$

- $S_0 = a \qquad S_n = S_n(x, a) = a + \log |x M_{1n}|$

$(X_n^\theta, S_n)_n$ semi-markovian random walk on $X \times \mathbb{R}$

$\qquad \dots$ centered when $\theta = \theta^*$

$\mathbb{P}^\theta$ corresponding probability $\qquad \mathbb{P}^0 = \mathbb{P}$ initial probability.

B. Fluctuations of (S_n)

$x \in X$, $a \in \mathbb{R}$ $\quad \tau_{x,a} = \inf\{ n \geq 1 : a + S_n \}$

Grama, Lauvergnat, Le Page
(Denisov-Wachtel's approach)

- $\mathbb{P}^{\theta^*}(\tau_{x,a} > n) \sim \dfrac{h^{\theta^*}(x,a)}{\sqrt{n}}$
- $\mathbb{P}^{\theta^*}(\tau_{x,a} > n, S_n(x,a) \in [b, b+h]) \leq \dfrac{C}{n^{3/2}} h^{\theta^*}(x,a) h'^{\theta^*}(x,b)$

$$\liminf_{+\infty} n^{3/2} \mathbb{P}^{\theta^*}(\tau_{x,a} > n, S_n \in [b, b+h]) \geq \frac{1}{C} h^{\theta^*}(x,a) h'^{\theta^*}(x,b)$$

Consequences $\quad m_n = \min(|M_1|, \dots, |M_{1n}|)$

For $K \subset \mathbb{R}$ compact and $a \geq 0$

- $\mathbb{P}(m_n \geq e^{-a}) \leq C \dfrac{\lambda(\theta^*)^n}{n^{3/2}} e^{\theta^* a}(1+a)$
- $\liminf_{+\infty} \dfrac{n^{3/2}}{\lambda(\theta^*)^n} \mathbb{P}(m_n \geq e^{-a}, |M_{1n}| \in K) \geq \dfrac{1}{C} e^{\theta^* a} h^{\theta^*}(x,a)$

C. Bounds for the survival probability

Upper bound

$$\mathbb{E}(q_n^{(i)}) \leq \underbrace{\mathbb{E}(q_n^{(i)}, m_n < 1)} + \mathbb{P}(m_n \geq 1)$$

$$\lesssim \sum_{a \geq 1} e^{-a}\, \mathbb{P}(e^{-a} \leq m_n < e^{-a+1})$$

$$\lesssim \sum_{a \geq 1} e^{-a}\, \mathbb{P}(m_n \geq e^{-a})$$

$$\lesssim \sum_{a \geq 1} e^{-(1-\theta^*)a}\, \frac{\lambda(\theta^*)^n}{n^{3/2}}$$

$$\leq C\, \frac{\lambda(\theta^*)^n}{n^{3/2}}$$

$\theta^* < 1$

Lower bound

Decompose $\frac{1}{|M_1|} + \dots + \frac{1}{|M_{1n}|}$ as $\underbrace{\sum_{1}^{k}}_{A} + \underbrace{\sum_{k+1}^{n-k}}_{B} + \underbrace{\sum_{n-k+1}^{n}}_{C}$

$$\mathbb{E}(q_n^{(1)}) \geq \mathbb{E}\left(\frac{1}{1+A+B+C} ; m_n \geq 1\right)$$

$$\geq \underbrace{\mathbb{E}\left(\frac{1}{1+A+C} ; m_n \geq 1\right)} - \mathbb{E}(B ; m_n \geq 1)$$

$$\geq \mathbb{E}\left(\frac{1}{1+A} ; m_{n-k} \geq 1\right) \times \mathbb{E}\left(\frac{1}{1+|M_1'|+\dots+|M_{k1}'|} ; |M_1'|, \dots |M_{k1}'| \leq 1\right)$$

$M_1' = M_n$

$\dots$

$M_k' = M_{n-k+1}$

$$\geq \frac{\lambda(\theta^*)^{n-k}}{n^{3/2}} a_*(k) \times c_*(k)$$

$$\mathbb{E}(B ; m_n \geq 1) \lesssim \frac{\lambda(\theta^*)^n}{n^{3/2}} \frac{1}{\sqrt{k}}$$

Letting $k \to +\infty$: $\liminf_k a_*(k), \lambda(\theta^*)^{-k} c_*(k) > 0.$

(Using the fact that $\sum_{n \geq 0} \underbrace{\mathbb{E}^{\theta^*}\left(|M_{1n}|^{-\lambda}, m_n \geq 1\right)}_{\lesssim 1/n^{3/2}} < +\infty$), $\lambda > 0$ ∎