The proportion between two CBI

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Objective: give models on the frequency of a type of individuals competting with other individuals, and find their genealogies by the use of duality and to find the corresponding coalescent processes.

- First case : the general model of two populations described by two indepenent CBI's.
- The two island model, or seedbank model
- A model for non independent CBI's.
- logistic growth case
- Discrete models.

First Model

We work with two independent Continuous-state branching process with immigration (CBI), each one has a branching mechanism ψ^i and an immigration mechanism φ^i, given for $i = 1, 2$ by

$$\psi^i(\lambda) = b^i\lambda + c^i\lambda^2 + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda x 1_{(0,1)}(x))m^i(dx), \qquad \lambda \geq 0$$

$$\varphi^i(\lambda) = \eta^i\lambda + \int_0^\infty (e^{-\lambda x} - 1)\nu^i(dx), \qquad \lambda \geq 0$$

- $b^i \in \mathbb{R}$, $c^i \geq 0$, and m^i is a measure concentrated on $(0, \infty)$ which satisfies that $\int_{(0,\infty)}(1 \wedge x^2)m^i(dx) < \infty$,
- $\eta^i \geq 0$ and ν^i is a measure on $(0, \infty)$ such that $\int_{(0,\infty)}(1 \wedge x)\nu^i(dx) < \infty$.

SDE

They satisfy the following SDE:

$$
\begin{aligned}
X_t^{(i)} = x^{(i)} &+ \int_0^t \sqrt{2c^i X_{s-}^{(i)}} dB_s^{(i)} - b^i \int_0^t X_s^{(i)} ds \\
&+ \int_0^t \int_{[1,\infty]} \int_0^{X_{s-}^{(i)}} z N^{(i)}(ds, dz, du) \\
&+ \int_0^t \int_{(0,1)} \int_0^{X_{s-}^{(i)}} z \tilde{N}^{(i)}(ds, dz, du) + \xi_t^{(i)}, \qquad t \geq 0, \qquad (1)
\end{aligned}
$$

The model

We consider the Markov process $(R_t, Z_t), t \geq 0$ with the same definitions

- $Z_t = X_t^{(1)} + X_t^{(2)}, \;\; t \geq 0$
- $R_t := \frac{X_t^{(1)}}{X_t^{(1)} + X_t^{(2)}}, \;\; t \geq 0$

But without the hypothesis Z_t =constant for al $t \geq 0$, and in this case, we do not have that $\{R_t : \;\; t \geq 0\}$ is markovian.

Using the Gillespie procedure (or culling procedure) we are able to find a Markov process representing the frecucny process as a solution of a SDE.

First definition of Λ-frecuency process

$$dR_t = R_{t-}(1-R_{t-})1_U\left(b^2-b^1+\frac{2}{z}(c^2-c^1)+\int\frac{w^2}{w+z}(m^2-m^1)(dw)\right)dt$$

$$[+\eta^1\frac{(1-R_{t-})}{z}-\eta^2\frac{R_{t-}}{z}]dt+\sqrt{\frac{1}{z}R_{t-}(1-R_{t-})[c^1(1-R_{t-})+c^2R_{t-}]}1_U dB_t$$

$$+\int_{(0,1)\times(0,\infty)} g^z(\circ)\tilde{N}_1(dt,dw,dv)+\int_{(0,1)\times(0,\infty)} h^z(\circ)\tilde{N}_2(dt,dw,dv)$$

$$+\int_{[1,\infty)\times(0,\infty)} g^z(\circ)N_1(dt,dw,dv)+\int_{[1,\infty)\times(0,\infty)} h^{(z)}(\circ)N_2(dt,dw,dv)$$

$$+\int_{(0,\infty)}\tilde{g}^{(z)}(\cdot)N_3(dt,dw)+\int_{(0,\infty)}\tilde{h}^{(z)}(\cdot)N_4(dt,dw)$$

$$R_0 = r.$$

and $(\circ)=(R_{t-},w,v)$, $(\cdot)=(R_{t-},w)$, $U=\{R_{t-}^{(z,r)}\in[0,1]\}$

First definition of Λ-frecuency process

with

(i) $B = \{B_t : t \geq 0\}$ is a standard Brownian motion.
(ii) $N_1(dt, dw, dv)$ is a PRM on $(0, \infty)^3$ with intensity measure $dtm^1(dw)dv$.
(iii) $N_2(dt, dw, dv)$ is a PRM on $(0, \infty)^3$ with intensity measure $dtm^2(dw)dv$.
(iv) $N_3(dt, dw)$ is a PRM $(0, \infty)^2$ with intensity measure $dt\nu^1(dw)$.
(v) $N_4(dt, dw)$ is a PRM on $(0, \infty)^2$ with intensity measure $dt\nu^2(dw)$.

all of them independent of each other.

And for $(x, w, v) \in (0, \infty)^3$,
$g_1^{(z)}(x, w, v) := \frac{w}{z+w}(1-x)1_{\{v \leq xz\}}, \;\; g_2^{(z)}(x, w, v) := -\frac{w}{z+w}x1_{\{v \leq (1-x)z\}}$.
For $(x, w) \in (0, \infty)^2$,
$\tilde{g}_1^{(z)}(x, w) := \frac{w}{z+w}(1-x), \;\; \tilde{g}_2^{(z)}(x, w) := -\frac{w}{z+w}x$.

First definition of Λ-frecuency process

Theorem

- *There exists a unique strong solution* $R^{(z,r)}$ *to the former SDE*
- $R_t^{(z,r)} \in [0,1]$ *for all* $t \geq 0$ $\mathbb{P}$ *a.s.*
- *for any* $t > 0$, *there exists a constant* $C(t) > 0$ *such that*

$$\boldsymbol{E}\left[|R_t^{(z,r)} - R_t^{(z,\overline{r})}|\right] \leq C(t)|r - \overline{r}|, \qquad r, \overline{r} \in [0,1].$$

- *it is a Feller process whose generator can be given explicitly.*

infinitesimal generator

The infinitesimal generator of $R^{(z,r)}$ is (for any $f \in \mathcal{C}^2([0,1])$)

$$\mathcal{L}^{(z)} f(r) = r(1-r)\left[f'(r)(b^2 - b^1 + \frac{2}{z}\left(c^2 - c^1\right)) + f''(r)\frac{c^1(1-r) + c^2 r}{z}\right]$$

$$+\frac{\eta^1}{z} f'\left(r\right)\left(1-r\right) + \int_{(0,1)} \left[f\left(r(1-u)+u\right) - f\left(r\right)\right] \mathbf{T}^{(\mathbf{z})}(\nu^1)(du)$$

$$-\frac{\eta^2}{z} f'\left(r\right) r + \int_{(0,1)} \left[f\left(r(1-u)\right) - f\left(r\right)\right] \mathbf{T}^{(\mathbf{z})}(\nu^2)(du).$$

$$+zr\int_{(0,1)} \left[f\left(r(1-u)+u\right) - f\left(r\right) - \frac{u}{1-u} f'\left(r\right)\left(1-r\right) 1_K\right] \mathbf{T}^{(\mathbf{z})}(m^1)(du)$$

$$+z(1-r)\int_{(0,1)} \left[f\left(r(1-u)\right) - f\left(r\right) + \frac{u}{1-u} f'\left(r\right) r 1_K\right] \mathbf{T}^{(\mathbf{z})}(m^2)(du)$$

Transformation

With the definition of $\mathbf{T}^{(\mathbf{z})}$ given by,

$$\mathbf{T}^{(\mathbf{z})} : \mathcal{M}[0,\infty] \mapsto \mathcal{M}[0,1]$$

where $\mathcal{M}[0,\infty]$ and $\mathcal{M}[0,1]$ denotes the space of measures in $[0,\infty]$ and $[0,1]$ respectively and for every measurable set $A \subset [0,1]$ and $\nu \in \mathcal{M}[0,\infty)$,

$$\mathbf{T}^{(\mathbf{z})}(\nu)(A) = \nu(T_z^{-1}(A))$$

with

$$T_z : [0,\infty) \mapsto [0,1], \text{given by } T_z(w) = w/(w+z).$$

In the case of two equally distributes CB, (with no immigration) we can be recover de generator of a Λ-frecuency process since in this case,

$$\mathcal{L}^{(z)} f(r) = \frac{c}{z} r(1-r) +$$

$$+ zr \int_{(0,1)} \left[f\left(r(1-u)+u\right) - f\left(r\right) - \frac{u}{1-u} f'\left(r\right)(1-r) 1_K \right] \mathbf{T}^{(\mathbf{z})}(m)(du)$$

$$+ z(1-r) \int_{(0,1)} \left[f\left(r(1-u)\right) - f\left(r\right) + \frac{u}{1-u} f'\left(r\right) r 1_K \right] \mathbf{T}^{(\mathbf{z})}(m)(du)$$

and this is equal to

$$\frac{c}{z} r(1-r) + z \int_{(0,1)} \left[r f(r(1-u)+u) + (1-r) f(r(1-u)) - f(r) \right] \mathbf{T}^{(\mathbf{z})}(m)(du)$$

Inspired by Gillespie's model , we would like to maintain the total size of the population constant, while allowing the frequency process R to evolve randomly; therefore, obtaining a one-dimensional stochastic process. Using this sampling method we obtain a stochastic model of the frequency of a particular type in the population under the assumption that the total population size is constant. The sequence of Markov jump processes $\{\overline{R}^{(z,n)} : n \geq 1\}$ converges weakly to the process $R^{(z,r)}$ given as the unique solution to our SDE, which (by construction) can be understood as having the same dynamics of the first coordinate of the process (R, Z) but with total population size constant and equal to $z > 0$.

Theorem

For any fixed $z > 0$ and $T > 0$, $\overline{R}^{(z,n)} \to R^{(z,r)}$ as $n \to \infty$ weakly in $\mathbb{D}([0, T], [0, 1])$.

Asymptotic behavior

We will study the asymptotic behavior of the Λ-asymmetric frequency process $R^{(z,r)}$ as the size of the population becomes large. To this end we introduce the deterministic process $R^{(\infty,r)} = \{R_t^{(\infty,r)} : t \geq 0\}$ given by

$$R_t^{(\infty,r)} = \frac{re^{\left(\psi^{(2)\prime}(0+)-\psi^{(1)\prime}(0+)\right)t}}{(1-r)+re^{\left(\psi^{(2)\prime}(0+)-\psi^{(1)\prime}(0+)\right)t}}, \qquad t \geq 0.$$

The large population limit of a Λ-asymmetric frequency process $R^{(z,r)}$ is given in the following result.

Theorem

Fix $T > 0$ and assume that $\int_{(1,\infty)} wm^{(i)}(dw) < \infty$ for $i = 1, 2$. Then

$$\lim_{z\to\infty} \boldsymbol{E}\left[\sup_{t\leq T} |R_t^{(z,r)} - R_t^{(\infty,r)}|^2\right] = 0.$$

For example in the case with no jumps, no immigration, and $c^{(1)} = c^{(2)}$, we obtain that

$$R_t^{(z,r)} - R_t^{(\infty,r)} = A_t^{(1,z)} + A_t^{(2,z)}, \qquad t \geq 0,$$

with

$$A_t^{(1,z)} := \int_0^t \sqrt{\frac{2}{z} R_{s-}^{(z,r)} (1 - R_{s-}^{(z,r)})} dB_s, \qquad t \geq 0,$$

$$A_t^{(2,z)} := \int_0^t (b^{(2)} - b^{(1)}) R_s^{(z,r)} (1 - R_s^{(z,r)}) ds -$$

$$- \int_0^t (b^{(2)} - b^{(1)}) R_s^{(\infty,r)} (1 - R_s^{(\infty,r)}) ds, \qquad t \geq 0.$$

The estimation for the term $A^{(1,z)}$ gives

$$\boldsymbol{E}\left[\left(\sup_{u \leq t} \int_0^u \sqrt{\frac{2}{z} R_{s-}^{(z,r)} (1 - R_{s-}^{(z,r)})} dB_s\right)^2\right] \leq$$

$$\frac{C_1}{z} \boldsymbol{E}\left[\int_0^T R_s^{(z,r)} (1 - R_s^{(z,r)}) ds\right] \leq \frac{C_1}{z} T,$$

and for $A^{(2,z)}$ we obtain that $\boldsymbol{E}\left[\left(\sup_{u\le t} A^{(2,z)}_u\right)^2\right] =$

$$=\boldsymbol{E}\left[\left(\sup_{u\le t}\left|\int_0^t (b^{(2)}-b^{(1)})R_s^{(z,r)}(1-R_s^{(z,r)})ds\right.\right.\right.$$

$$\left.\left.\left.-\int_0^t (b^{(2)}-b^{(1)})R_s^{(\infty,r)}(1-R_s^{(\infty,r)})ds\right|\right)^2\right]$$

$$\le C_2 T\,\boldsymbol{E}\left[\int_0^t \sup_{u\le s}|R_u^{(z,r)}-R_u^{(\infty,r)}|^2 du\right], \qquad t\in[0,T].$$

Hence

$$\boldsymbol{E}\left[\sup_{u\le t}|R_u^{(z,r)}-R_u^{(\infty,r)}|^2\right] \le \frac{C_1}{z}T + C_2 T\,\boldsymbol{E}\left[\int_0^t \sup_{u\le s}|R_u^{(z,r)}-R_u^{(\infty,r)}|^2 du\right].$$

Finally, an application of Gronwall's inequality, we obtain that for $T>0$

$$\boldsymbol{E}\left[\sup_{u\le T}|R_u^{(z,r)}-R_u^{(\infty,r)}|^2\right] \le \frac{C_1}{z}T e^{C_2 T}.$$

This proves the result in this special case.

Large population limit

We have defined

$$R_t^{(\infty,r)} = \frac{re^{\left(\psi^{(2)\prime}(0+)-\psi^{(1)\prime}(0+)\right)t}}{(1-r)+re^{\left(\psi^{(2)\prime}(0+)-\psi^{(1)\prime}(0+)\right)t}}, \qquad t \geq 0,$$

which is solution to the logistic equation given by

$$dR_t^{(\infty,r)} = (\psi^{(2)\prime}(0+)-\psi^{(1)\prime}(0+))R_t^{(\infty,r)}(1-R_t^{(\infty,r)})dt, \qquad t > 0,$$

with initial condition $R_0^{(\infty,r)} = r$.

Large population limit and fluctuation of the Λ AFP

Theorem

If $\int_{(0,\infty)} w^2 m^{(i)}(dw) + \int_{[1,\infty)} w\nu^{(i)}(dw) < \infty$ *for* $i = 1, 2$ *then*

- *for any fixed* $T > 0$,

$$\sqrt{z}(R^{(z,r)} - R^{(\infty,r)}) \to e^{-U} X^{(\infty)}, \;\; z \to \infty$$

  weakly in $\mathbb{D}([0,T], \mathbb{R})$, Where :

1.- $X^{(\infty)}$ is a zero mean Gaussian process with covariance function $C_{X^{(\infty)}}$

2.- For $s, t \geq 0$,

$$C_{X^{(\infty)}}(s,t) := \int_0^{s\wedge t} e^{2U_u} R_u^{(\infty,r)}(1-R_u^{(\infty,r)})[\sigma^{(1)}(1-R_u^{(\infty,r)})+\sigma^{(2)} R_u^{(\infty,r)}]du$$

3.-$\sigma^{(i)} = 2c^{(i)} + \int_{(0,\infty)} w^2 m^{(i)}(dw).\ \ i = 1, 2,$

4.-$U_t := \left(\psi^{(2)\prime}(0+) - \psi^{(1)\prime}(0+)\right) \int_0^t (2R_s^{(\infty,r)} - 1)ds, \qquad t \geq 0.$

Note that we can represent $X^{(\infty)}$ as a time-changed Brownian motion, i.e.

$$X^{(\infty)} \overset{\mathcal{L}}{=} \left\{ W_{\int_0^t C_{X^{(\infty)}}(s,s)ds}; t \geq 0 \right\},$$

where W is Brownian motion.

Ideas of the proof. We define

$$dX_t^{(z)} = e^{U_t}\sqrt{2R_t^{(\infty,r)}(1-R_t^{(\infty,r)})[c^{(1)}(1-R_t^{(\infty,r)})+c^{(2)}R_t^{(\infty,r)}]}dB_t$$
$$+\sqrt{z}\int_{(0,\infty)^2} e^{U_t}g^{(z)}(R_t^{(\infty,r)},w,v)\tilde{N}_1(dt,dw,dv)$$
$$+\sqrt{z}\int_{(0,\infty)^2} e^{U_t}h^{(z)}(R_t^{(\infty,r)},w,v)\tilde{N}_2(dt,dw,dv).$$

Result 1.-If $X^{(z_n)} \to Y^{(\infty)}$ as $z_n \to \infty$ weakly in $\mathbb{D}([0,T],\mathbb{R})$.
Then, $Y^{(\infty)} \stackrel{\mathcal{L}}{=} X^{(\infty)}$ as elements of $\mathbb{D}([0,T],\mathbb{R})$

Result 2 : the family $\{\sqrt{z}(R^{(z,r)} - R^{(\infty,r)}) : z \geq 1\}$ is tight in $\mathbb{D}(\mathbb{R}_+,\mathbb{R})$

The result 1, is proved by a characterization of the finite dimensional distributions using characteristic functions.

$$\Phi(\lambda, z) = \boldsymbol{E}\left[e^{i\lambda \sum_{i=1}^{n} a_i (X^{(z)}(t_i) - X^{(z)}(t_{i-1}))}\right], \qquad \lambda \in \mathbb{R}\,.$$

Using the fact that B, $\tilde{N}_1(dt, du, dv)$, and $\tilde{N}_2(dt, du, dv)$ are independent, we obtain for $\lambda \in \mathbb{R}$

$$\Phi(\lambda, z) = \Phi_1(\lambda) \prod_{i=2}^{3} \Phi_i(\lambda, z)$$

as $(z \to \infty)$ converges to

$$\prod_{i=1}^{n} \exp\left\{-\frac{(\lambda a_i)^2}{2} \int_{t_{i-1}}^{t_i} e^{2U_t} R_t^{(\infty,r)} (1 - R_t^{(\infty,r)})\right.$$

$$[\sigma^{(1)}(1 - R_t^{(\infty,r)}) + \sigma^{(2)} R_t^{(\infty,r)}]dt, \;\; z \to \infty$$

Result 2, allows us to say that there exists a subsequence $\{z_n\}_{n\geq 1}$ such that $\{(\sqrt{z_n}(R_t^{(z_n,r)} - R_t^{(\infty,r)}))_{t\geq 0} : n \geq 1\}$ converges weakly to some $(Y^{(\infty)})_{t\geq 0}$ in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$.

With some extensive calculations we obtain

$$X_t^{(z_n)} = e^{U_t}\sqrt{z_n}(R_t^{(z_n,r)} - R_t^{(\infty,r)}) - \sqrt{z_n}A_t^{(5,z_n)} + A_t^{(6,z_n)} - (\sqrt{z_n}A_t^{(4,z_n)} - X_t^{(z_n)}$$

So $e^U Y^{(\infty)} \overset{\mathcal{L}}{=} X^{(\infty)}$, which implies that the limit of any convergent subsequence of the family $\{\sqrt{z}(R^{(z,r)} - R^{(\infty,r)}) : z \geq 1\}$ is equal in law to $e^{-U}X^{(\infty)}$. Therefore,

$$\sqrt{z}(R^{(z,r)} - R^{(\infty,r)}) \to e^{-U}X^{(\infty)}, \qquad \text{as } z \to \infty,$$

weakly in $\mathbb{D}([0,T], \mathbb{R})$.

Coalescents and frequency processes

- The celebrated seven-authors paper, (Birkner,Blath,Capaldo, Etherige.Moehle, Schweinsberg,Wakolbinger) considers the ratio of two independent identically distributed α-stable CB processes and by a time change they obtain the best possible relation with coalescents, $\tau(t) = \int_0^t \frac{ds}{Z_s^{1-\alpha}}$ then $X_t = Z_{\tau^{-1}(t)}$ is the moment dual of the block counting process of a coalescent,
- Johnston and Lambert studied the genealogy of general CB processes and discovered that, although the genealogy is not a Markovian object in general, it can be coupled to Λ-coalescents at small times.
- Bertoin and Le Gall, observed that the Bolthausen Szmitman coalescent describes the genealogy of Neveu's CB process. This is done in one of the three seminal papers, dealing with this subject, and where the Flow of bridges is introduced. They study the genealogy associated to the Neveu' CB process (neither by duality or time change.....)

Given a finite measure Λ on $[0,1]$ the block counting process of a Λ-coalescent, $N = \{N_t : t \geq 0\}$, is an $\mathbb{N}$-valued decreasing process that goes from the state n to the state $n-i+1$, for any $i \in \{2, ..., n\}$ at rate $\binom{n}{i}\lambda_{n,i}$, where

$$\lambda_{n,i} := \int_0^1 y^i(1-y)^{n-i}\frac{\Lambda(dy)}{y^2}.$$

These processes have a biological interpretation: they are related to the genealogy of a population (in a generalized Wright-Fisher model) and they are moment duals to frequency processes.

Coalescents and frequency processes

They are called frequency processes because they arise as scaling limits of the frequency of individuals of a certain type in a generalized Wright-Fisher model. In light of these facts, moment duality relates the genealogy of a population with the evolution of its genetic profile.
To say that F and N are moment duals is equivalent to saying that for all $x \in [0,1]$, $n \in \mathbb{N}$ and $t > 0$

$$\boldsymbol{E}_x[(F_t)^n] = \boldsymbol{E}_n[x^{N_t}].$$

Moment duality for the Λ-asymmetric frequency process.

- We will show the relationship between the Λ-asymmetric frequency process $R^{(z,r)}$, and a particular class of branching-coalescent processes.
- This class consists of continuous time Markov chains taking values in $\mathbb{N}_0 \cup \{\Delta\}$ (where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$). The point Δ is a cementery state and we assume that $x^{\Delta} = 0$ for all $x \in [0,1]$.
- For each $i, k \in \mathbb{N}_0$ with $i \geq k$, and $v \in [0,1]$ we define the following terms

$$\lambda^l_{i,k}(v) = \int_{(0,v)} \left[(1-u)^{i-k} u^k\right] \mathbf{T}^{(\mathbf{z})}(m^l)(du),$$

$$\mu^l_{i,k} = \int_{(0,1)} \left[(1-u)^{i-k} u^k\right] \mathbf{T}^{(\mathbf{z})}(\nu^l)(du), \qquad l = 1, 2.$$

Moment duality for the Λ-asymmetric frequency process.

Now for each $i, j \in \mathbb{N}_0 \cup \{\Delta\}$ let us consider the following set of real numbers

$$q^z_{ij} = \begin{cases} \overline{\mu}^1_{i,i} & \text{if } i \in \mathbb{N} \text{ and } j = 0, \\ si + \sum_{k=2}^{i} \kappa_k \binom{i}{k} + \beta_i & \text{if } i \in \mathbb{N} \text{ and } j = i+1, \\ \binom{i}{i-j} \overline{\mu}^1_{i,i-j} + \binom{i}{i-j+1} \overline{\lambda}^1_{i,i-j+1} & \text{if } i \geq 2 \text{ and } j \in \{1, .., i-1\}, \\ \alpha_i & \text{if } i \in \mathbb{N} \text{ and } j = \Delta, \\ 0 & \text{otherwise.} \end{cases}$$

Moment duality for the Λ-asymmetric frequency process.

where

- For $2 \leq k \leq i$, $\overline{\lambda}^1_{i,k} = \int_{[0,1)} \left[(1-u)^{i-k}u^k\right] u^{-2}\Lambda^1(du)$ and

$$\Lambda^1(du) = \frac{c^1}{z}\delta_0(du) + zy^2\mathbf{T}^{(\mathbf{z})}(m^1)(du)$$

- For $1 \leq k \leq i$, $\overline{\mu}^1_{i,k} = \int_{[0,1)} \left[(1-u)^{i-k}u^k\right] u^{-1}\Gamma^1(du)$, with

$$\Gamma^1(du) = \frac{\eta^1}{z}\delta_0(du) + y\mathbf{T}^{(\mathbf{z})}(\nu^1)(du)$$

.

Moment duality for the Λ-asymmetric frequency process.

$$s = \frac{2(c^1 - c^2)}{z} + (b^1 - b^2) -$$

$$z\left[\left(\lambda_{i,1}^{1}(1) - \lambda_{i,1}^{2}(1)\right) - \left(\lambda_{i,1}^{1}\left(\frac{1}{1+z}\right) - \lambda_{i,1}^{2}\left(\frac{1}{1+z}\right)\right)\right]$$

$$+z\left(\int_{(0,1/(1+z))} \frac{u^2}{1-u^2} m^1 \circ T^{-1}(du) - \int_{(0,1/(1+z))} \frac{u^2}{1-u^2} m^2 \circ T^{-1}(du)\right).$$

Moment duality for the Λ-asymmetric frequency process.

For $k \geq 1$

$$\kappa_k = z\left[k\left(\lambda^1_{i,k}\left(\frac{1}{1+z}\right) - \lambda^2_{i,k}\left(\frac{1}{1+z}\right)\right) - \left(\lambda^1_{i,k}(1) - \lambda^2_{i,k}(1)\right)\right] +$$

$$+\frac{2(c^1 - c^2)}{z} 1_{\{k=2\}}$$

$$\beta_k = -kz\left[\left(\lambda^1_{k,1}(1) - \lambda^2_{k,1}(1)\right) - \left(\lambda^1_{k,1}\left(\frac{1}{1+z}\right) - \lambda^2_{k,1}\left(\frac{1}{1+z}\right)\right)\right].$$

$$\alpha_k = \int_{[0,1)} (1-(1-u)^k)\rho^2(du),$$

with $\rho^2(du) = \frac{\eta^2}{z}\delta_0(du) + y\mathbf{T}^{(\mathbf{z})}(\nu^2)(du)$.

All these quantities can be interpreted as a notion of generalized ancestry in presence of possibly skewed and asymmetric reproduction mechanisms, population dependent variance, mutation, coordinated mutation, and selection.

When the two CBI Processes $X^{(1)}$ and $X^{(2)}$ are v equally distributed CB with characteristic triplet (b^1, c^1, m^1) and $\xi^{(i)} = 0$ for $i = 1, 2$.

$$\mathcal{L}^{(z)} f(r) = c^1 \frac{r(1-r)}{z} f''(r) +$$

$$+z \int_{(0,1)} \Big[r f\left(r(1-u)+u\right) + (1-r) f\left(r(1-u)\right) - f\left(r\right) \Big] \mathbf{T}^{(\mathbf{z})}(m^1)(du),$$

so $R^{(z,r)}$ corresponds to the classic Λ-frequency process, whose dual is the block counting process of a Λ-coalescent. Indeed, by the duality theorem we have that the associated moment dual $N^{(z,r)}$ has a generator $Q^z = (q^z_{ij})_{i,j\in\mathbb{N}}$ given by

$$q_{ij} = \begin{cases} \binom{i}{i-j+1} \overline{\lambda}^{1}_{i,i-j+1} & \text{if } i \geq 2 \text{ and } j \in \{1, .., i-1\}, \\ 0 & \text{otherwise,} \end{cases}$$

where for $2 \leq k \leq i$,

$$\overline{\lambda}^1_{i,k} = \int_{[0,1)} \left[(1-u)^{i-k} u^k \right] u^{-2} \Lambda^1(du),$$

and $\Lambda^1(du) = \frac{c^1}{z} \delta_0(du) + zy^2 \mathbf{T}^{(\mathbf{z})}(m^1)(du)$.

A metric on the class of CB processes

For each triplet $(b, c, m) \in \mathbb{R} \times \mathbb{R}_+ \times \mathcal{LM}(\mathbb{R}_+)$ we recall the branching mechanism ψ

$$\psi(\lambda) = b\lambda + c\lambda^2 + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda x 1_{(0,1)}(x)) m(dx), \qquad \lambda \geq 0$$

An equivalent form is,

$$\psi(\lambda) = \tilde{b}\lambda + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \frac{\lambda x}{1+x^2}) \frac{x^2+1}{x^2} \tilde{m}(dx), \qquad \lambda \geq 0$$

$$\tilde{b} := b + \int_{\mathbb{R}_+ \setminus \{0\}} \left(\frac{x}{x^2+1} - x 1_{\{|x| \leq 1\}} \right) m(dx),$$
$$\tilde{m} := \sigma^2 \delta_0(dx) + \frac{x^2}{x^2+1} m(dx).$$

So, in the space of triplets $\mathbb{R} \times \mathbb{R}_+ \times \mathcal{LM}(\mathbb{R}_+)$ we can introduce the following metric:

$$d_\psi((b^{(1)}, c^{(1)}, m^{(1)}), (b^{(2)}, c^{(2)}, m^{(2)})) := |\tilde{b}^{(1)} - \tilde{b}^{(2)}| + \rho(\tilde{m}^{(1)}, \tilde{m}^{(2)}),$$

where ρ denotes the Prohorov distance in the space of finite measures (the former transformation, allows us to work with a finite measures $\tilde{m}$.)

Proposition

Let $\{Z^n\}$ be a sequence of continuous-state branching processes with the characteristic triplets (b_n, c_n, m_n). Additionally, consider a continuous-state branching process Z with the characteristic triplet (b, c, m). Assume that

$$\lim_{n\to\infty} d_\Psi((b_n, c_n, m_n), (b, c, m)) = 0.$$

Then $Z^n \to Z$ as $n \to \infty$, weakly on the space of cadlag paths from $\mathbb{R}_+$ to $[0, \infty]$ with the Skorohod topology if the branching mechanism ψ of Z is nonexplosive, and with the uniform Skorohod topology if ψ is explosive.

The proof, uses some convergence results , part of them g' in Kallenber's book in order to obtain convergence of the corresponding Lévy processes , and then we use the Lamperti's time change (between LP and CB processes), to get the convergence for the CB' s.

For our next result, we denote the space of finite measures on $[0,1]$ by $\mathcal{M}_F([0,1])$ and let $(\mathcal{P}, d)$ be the space of partitions of the natural numbers endowed with the distance d, which is defined for any two partitions $\pi, \pi' \in \mathcal{P}$ by

$$d(\pi, \pi') = M^{-1} \text{ if and only if } \pi|_{[M]} = \pi'|_{[M]} \text{ and } \pi|_{[M+1]} \neq \pi'|_{[M+1]}$$

where $\pi|_{[M]}$ is the restriction of π to $[M] = \{1, 2, ..., M\}$.

Proposition

Let $\{\Pi^N\}_{N\in\mathbb{N}_0}$ be a sequence of Λ-coalescents with characteristic measures $\{\Lambda^N\}_{N\in\mathbb{N}_0} \subset \mathcal{M}_F([0,1])$, such that $\Lambda^N \to \Lambda^0$ weakly as $N \to \infty$. Let Π^0 be the Λ-coalescent associated to Λ^0. Then

$$\Pi^N \to \Pi^0, \qquad \text{as } N \to \infty,$$

weakly in $\mathbb{D}(\mathbb{R}_+, (\mathcal{P}, d))$.

Theorem

Consider the metric space **L** *of* Λ*-coalescents with no atom at* $\{1\}$ *equipped with the Prohorov distance over the space of probability measures defined on the space* $\mathbb{D}(\mathbb{R}_+, (\mathcal{P}, d))$*. In addition, for* $r \in \mathbb{R}$*, consider the space* $\boldsymbol{\Psi}_r \subset \boldsymbol{\Psi}$ *of CB processes with* $\tilde{b} = r$ *equipped with the Prohorov distance over the space of probability measures defined on the space* $\mathbb{D}([0, T], \mathbb{R}_+)$ *endowed with the uniform Skorohod topology. Then,* **L** *and* Ψ_r *are homeomorphic.*
Furthermore, consider the mapping $\mathbf{H}^{(\mathbf{z})} : \boldsymbol{\Psi}_r \mapsto \mathbf{L}$ *such that a CB process with the triplet* (b, c, ν) *is mapped to the* Λ*-coalescent with the associate measure*

$$\mathbf{H}^{(\mathbf{z})}((b, c, \nu)) = \frac{c}{z}\delta_0 + zy^2\mathbf{T}^{(\mathbf{z})}(\nu).$$

Then, for every $z > 0$*,* $\mathbf{H}^{(\mathbf{z})}$ *is a homeomorphism, with inverse* ${\mathbf{H}^{(\mathbf{z})}}^{-1}$ *.*

It sends a Λ-coalescent to the CB process with characteristic triplet

- $(r - \int_{\mathbb{R}\setminus\{0\}} \left(\frac{x}{x^2+1} - x1_{\{|x|\leq 1\}}\right) \nu(dx))$
- $(z\Lambda(\{0\}))$,
- $(zy^2)^{-1}(\mathbf{T^{(z)}})^{-\mathbf{1}}(\Lambda - \Lambda(\{0\})\delta_0))$

THANK YOU FOR YOUR TIME AND ATTENTION.