The multitype bisexual Galton-Watson branching process

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Introduction

Two-sex branching processes were introduced by Daley in [2] and consist, at time $n \in \mathbb{N}$, of two disjoint classes (F_n and M_n) that form couples (Z_n) that accomplish reproduction

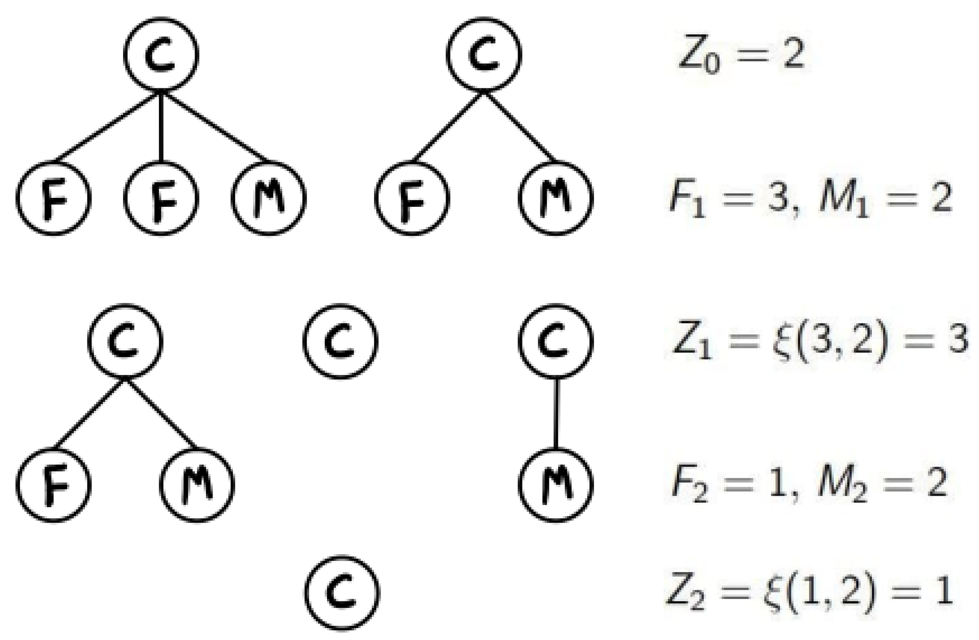

Figure 1: First two iterations of a bisexual GW process for some mating function ξ.

Since Daley's work, different models of two-sex population have been studied and, in particular, conditions for extinction and a malthusian behavior for this kind of processes have been established for general models. See for instance [1, 3, 5] and the references therein.

Our focus of study is the multitype bisexual process with a finite number of types. Although specific models were studied in the two-sex population literature, no general mathematical description for this kind of processes has yet been established.

We introduce the processes of females $(F_n)_{n\in\mathbb{N}}$, males $(M_n)_{n\in\mathbb{N}}$ and couples $(Z_n)_{n\in\mathbb{N}}$ taking values on $\mathbb{N}^{n_f}$, $\mathbb{N}^{n_m}$ and $\mathbb{N}^p$ respectively. We also consider a determistic *mating function* $\xi : \mathbb{N}^{n_m} \times \mathbb{N}^{n_m} \longrightarrow \mathbb{N}^p$

Assumption on ξ

We suppose that the mating function ξ is superadditive, which means that for all $x_1, x_2 \in \mathbb{N}^{n_f}$ and all $y_1, y_2 \in \mathbb{N}^p$,

$$\xi(x_1 + x_2, y_1 + y_2) \geq \xi(x_1, y_1) + \xi(x_2, y_2),$$

where the inequality holds componentwise.

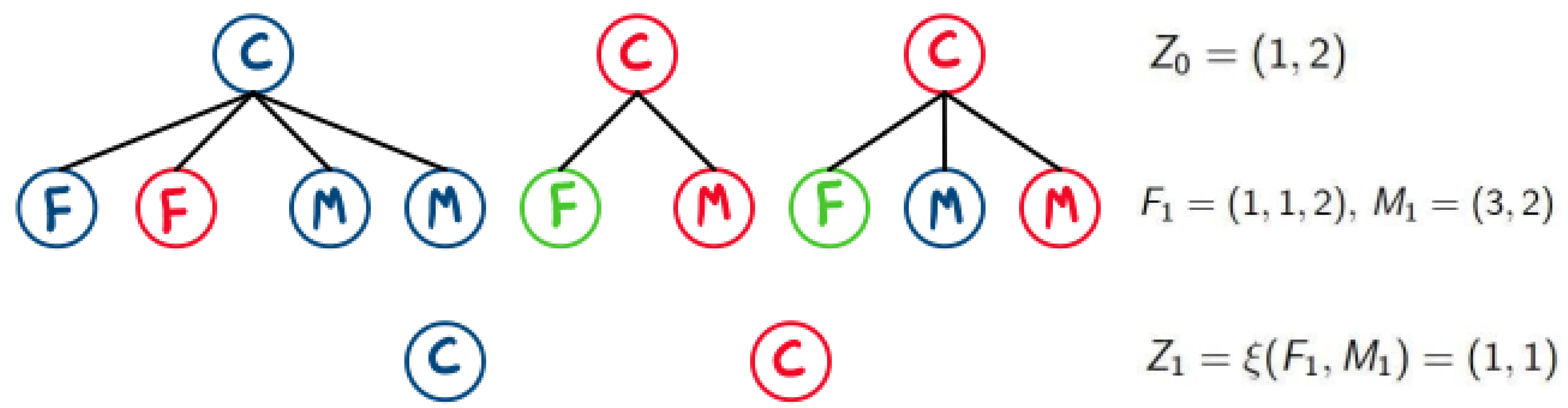

Figure 2: First itereation of a multitype bisexual GW process for some mating function ξ.

The R function

We define the function $R : \mathbb{R}_+^p \longrightarrow (\mathbb{R}_+ \cup \{+\infty\})^p$ by

$$R(z) = \lim_{k\to+\infty} \frac{\mathbb{E}(Z_1 \mid Z_0 = \lfloor kz \rfloor)}{k}, \qquad (1)$$

which is well defined thanks to the superadditivity of ξ. This ensures in addition that R is a concave function.

Eigenelements (see [4])

Assume that the function R defined in (1) is finite and suppose that $R^n(z) > 0$ for all n big enough and all $z \in \mathbb{R}_+^p \setminus \{0\}$. Then, the problem

$$R(z^*) = \lambda^* z^* \qquad (2)$$

has a unique solution with $\lambda^* > 0$ and $z^* > 0$, with $\|z\| = 1$.

Theorem: Extinction Conditions

Assume that $(Z_n)_{n\in\mathbb{N}}$ is a transitive process. Set R as in (1) and assume it is finite and the conditions for (2) hold. Then,

$$\mathbb{P}\left(Z_n \xrightarrow{n\to+\infty} 0 \mid Z_0 = z\right) = 1, \text{ for all } z \in \mathbb{N}^p \iff \lambda^* \leq 1.$$

If $\lambda^* > 1$ or if there exists $z' \in \mathbb{N}^p$ such that one of the components of $R(z')$ is not finite, then there exists $r > 0$ such that, if $|z| > r$, then $\mathbb{P}(Z_n \to 0 \mid Z_0 = z) < 1$.

Theorem: Asymptotic Profile

Assume the same conditions as in the previous Theorem and suppose R is finite. Then, for all $z \in \mathbb{N}^p$, there exists a real non-negative random variable $\mathcal{C}$ such that

$$\frac{Z_n}{(\lambda^*)^n} \xrightarrow[n\to+\infty]{\mathbb{P}(\cdot|Z_0=z)\ a.s.} \mathcal{C}z^*, \quad \frac{F_n}{(\lambda^*)^n} \xrightarrow[n\to+\infty]{\mathbb{P}(\cdot|Z_0=z)\ a.s.} \frac{1}{\lambda^*}\mathcal{C}z^*\mathbb{F} \text{ and } \frac{M_n}{(\lambda^*)^n} \xrightarrow[n\to+\infty]{\mathbb{P}(\cdot|Z_0=z)\ a.s.} \frac{1}{\lambda^*}\mathcal{C}z^*\mathbb{M}, \qquad (3)$$

with λ^* and z^* given by (2) and where $\mathbb{F}$ and $\mathbb{M}$ are explicit matrices.

Assume in addition that $\mathcal{C}$ is non-degenerate at 0. Then, for all $z \in \mathbb{N}^p$ and up to a $\mathbb{P}(\cdot \mid Z_0 = z)$ negligible event,

$$\{\mathcal{C} = 0\} = \{\exists n \in \mathbb{N},\ Z_n = 0\}.$$

Example: Perfect Fidelity

Set $n_m = n_f = p$ and consider the mating function

$$\xi(x,y) = (\min\{x_i, y_i\})_{1\leq i\leq p}.$$

If in addition we suppose that every new individual is a female with probability $\alpha \in (0,1)$, independent of its type, then

$$R(z) = \min\{\alpha, 1-\alpha\}\mathbb{U}z,$$

for $\mathbb{U}$ an explicit matrix.

Hence, if we assume that $\mathbb{U}^n > 0$ for all n big enough, we have

$$\lambda^* = \min\{\alpha, 1-\alpha\}\lambda_{\mathbb{U}} \text{ and } z^* = z_{\mathbb{U}},$$

with $\lambda_{\mathbb{U}}$ and $z_{\mathbb{U}}$ the eigenelements of $\mathbb{U}$.

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