Right-Most Position of a Last Progeny Modified Branching Random Walk

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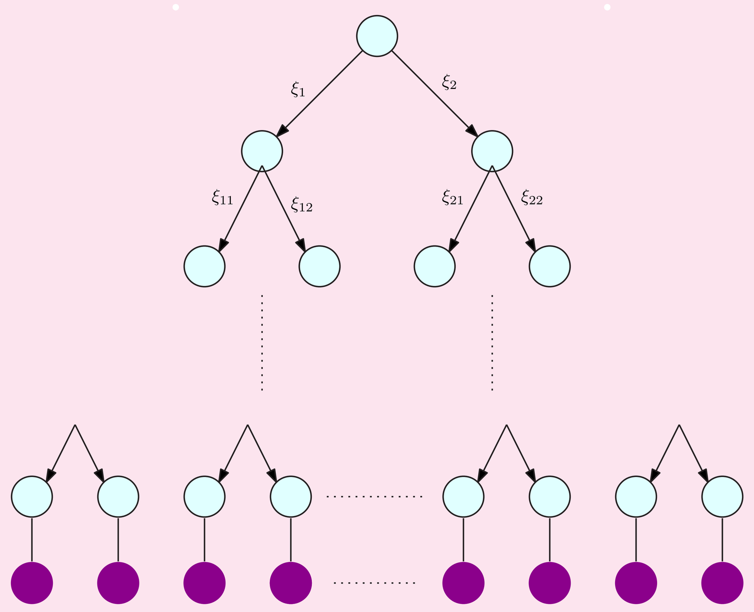
The Model

The time is discrete and starts at 0. At time $n = 0$, there is only one particle at the origin. At time $n = 1$, the particle dies and gives birth to some random number of offspring, which are then displaced according to a point process, say, $\mathbf{Z}$. Then, at each time $n \geq 2$, each particle at generation $n-1$ dies and produces some random number of offspring, which are then given random displacements independently and according to a copy of the point process $\mathbf{Z}$.



Once this has been done, all particles at generation n are given further displacements by a set of i.i.d. random variables, of the form $(\frac{1}{\theta}A_v)_{|v|=n}$, which are independent of the process so far. The distribution of this displacement variables, namely A_v's and the "scaling" constant $\theta > 0$ are two parameters of this model.

This new process will be called <u>L</u>*ast* <u>P</u>*rogeny* <u>M</u>*odified* <u>B</u>*ranching* <u>R</u>*andom* <u>W</u>*alk* (LPM-BRW).

If we denote by S_v the position of an individual v, then the right-most position of the classical branching random walk

$$R_n := \max_{|v|=n} S_v,$$

and the right-most position of the modified branching random walk

$$R_n^* := \max_{|v|=n} \left(S_v + \frac{1}{\theta} A_v \right).$$

Question: How far the family may have spread in generation n for large n ? In other words, what is the asymptotic distribution of R_n^* ?

Transforming Relation

In our model, we take a specific type of distribution for the A_v's. We take $A_v = \log \frac{Y_v}{E_v}$, where $(Y_v)_{|v|=n}$ are i.i.d. with some positively supported measure, say, μ and $(E_v)_{|v|=n}$ are i.i.d. Exponential(1) random variables and both sets of random variables are independent.

Theorem 1 (Basic Transforming Relation)

$$\max_{i \geq 1} \left(X_i + \log \frac{Y_i}{E_i} \right) \stackrel{d}{=} \log \left(\sum_{i \geq 1} e^{X_i} Y_i \right) - \log E,$$

where $\{Y_i\}_{i \geq 1}$ are i.i.d. positive and $E, \{E_i\}_{i \geq 1}$ are i.i.d. Exponential(1) and these sequences are independent of each other and also Independent of $\{X_i\}_{i \geq 1}$.

Theorem 2 (General Transforming Relation)

$$R_n^* \stackrel{d}{=} \frac{1}{\theta} \left(\log \left(\sum_{|v|=n} e^{\theta S_v} Y_v \right) - \log E \right).$$

Assumptions

Define

$$\nu(\theta) = \log \mathbb{E} \left[\int e^{\theta x} \, \mathbf{Z}(dx) \right]$$

We assume that $\nu(\theta) < \infty$ for all $\theta \in (-\vartheta, \infty)$ and $\nu(0) > 0$. Also, $\mathbb{E}[\mathbf{Z}(\mathbb{R})^{1+p}] < \infty$ for some $p > 0$.

Let θ_0 be such that the tangent to the graph of ν at the point $(\theta_0, \nu(\theta_0))$ passes through the origin.

SLLN for R_n^*

Theorem 3 *For every non-negatively supported probability $\mu \neq \delta_0$ that admits a finite mean*

$$\frac{R_n^*}{n} \xrightarrow{\text{a.s.}} \begin{cases} \frac{\nu(\theta)}{\theta} & \text{if} \quad \theta < \theta_0; \\ \frac{\nu(\theta_0)}{\theta_0} & \text{if} \quad \theta \geq \theta_0. \end{cases}$$

Comparison (Biggins[1976])

$$\frac{R_n}{n} \xrightarrow{\text{a.s.}} \frac{\nu(\theta_0)}{\theta_0}.$$

Centered asymptotic limits

Boundary Case ($\theta = \theta_0$) :

Theorem 5 *Assume that μ admits a finite mean, then there exists a random variable $H_{\theta_0}^\infty$ such that*

$$R_n^* - \frac{\nu(\theta_0)}{\theta_0} n + \frac{1}{2\theta_0} \log n \xrightarrow{d} H_{\theta_0}^\infty + \frac{1}{\theta_0} \log \langle \mu \rangle,$$

where $H_{\theta_0}^\infty = \frac{1}{\theta_0} \left[\log D_\infty - \log E + \frac{1}{2} \log \left(\frac{2}{\pi \sigma^2} \right) \right]$; D_∞ is the a.s. limit of the derivative martingale, *$D_n = -\frac{1}{m(\theta_0)^n} \sum_{|v|=n} (\theta_0 S_v - n\nu(\theta_0)) e^{\theta_0 S_v}$; and $E \sim$ Exponential(1) is independent of D_∞. Further, $\sigma^2 := \mathbb{E} \left[\frac{1}{m(\theta_0)^n} \sum_{|v|=n} (\theta_0 S_v - n\nu(\theta_0))^2 e^{\theta_0 S_v} \right]$.*

Below the Boundary ($\theta < \theta_0$) :

Theorem 6 *Assume that μ admits finite mean, then*

$$R_n^* - \frac{\nu(\theta)}{\theta} n \xrightarrow{d} H_\theta^\infty + \frac{1}{\theta} \log \langle \mu \rangle.$$

Above the Boundary ($\theta > \theta_0$) :

Theorem 7 *Suppose $\mu = \delta_1$, then*

$$R_n^* - \frac{\nu(\theta_0)}{\theta_0} n + \frac{3}{2\theta_0} \log n \xrightarrow{d} H_\theta^\infty.$$

Comparison (Aïdékon [2013]) *There exists a random variable H_∞ such that*

$$R_n - \frac{\nu(\theta_0)}{\theta_0} n + \frac{3}{2\theta_0} \log n \xrightarrow{d} H_\infty,$$

where $H_\infty = \frac{1}{\theta_0} [\log D_\infty - \log E + C]$ and C is a constant.

Brunet-Derrida Type Results

Theorem 4 *For any $\theta \leq \theta_0$,*

$$\sum_{|v|=n} \delta_{\{\theta S(v) - \log E_v - \theta R_n^*\}} \xrightarrow{d} \overline{\mathcal{Y}},$$

where $\mathcal{Y}$ is an inhomogeneous Poisson point process on $\mathbb{R}$ with intensity $e^{-x}\,dx$ and $\overline{\mathcal{Y}}$ is the point process $\mathcal{Y}$ seen from its right-most position.

Comparison (Madaule [2017])

$$\sum_{|v|=n} \delta_{\{\theta_0 S(v) - \theta_0 R_n\}} \xrightarrow{d} \overline{\mathcal{X}}$$

where $\mathcal{X}$ is a decorated Poisson point process on $\mathbb{R}$ and $\overline{\mathcal{X}}$ is the point process $\mathcal{X}$ seen from its right-most position.

Proof of Transforming Relation

$$\begin{aligned} & \max_{i \geq 1} \left(X_i + \log \frac{Y_i}{E_i} \right) \\ = & \max_{i \geq 1} \left(\log \frac{e^{X_i} Y_i}{E_i} \right) \\ = & -\log \left(\min_{i \geq 1} \frac{E_i}{e^{X_i} Y_i} \right) \\ \stackrel{d}{=} & -\log \frac{E}{\sum_{i \geq 1} e^{X_i} Y_i} \\ = & \log \left(\sum_{i \geq 1} e^{X_i} Y_i \right) - \log E. \end{aligned}$$

References

1. A. Badyopadhyay and P. P. Ghosh. Right-most position of a last progeny modified branching random walk. *arXiv:2106.02880, 2021.*
2. A. Badyopadhyay and P. P. Ghosh. Right-most position of a last progeny modified time inhomogeneous branching random walk. *Statistics & Probability Letters, 193: Paper No. 109697 (2023).*
3. P. P. Ghosh. Large deviations for the right-most position of a last progeny modified branching random walk. *Electronic Communications in Probability, 27: Paper No. 6 (2022).*