INSTITUT de MATHEMATIQUES de TOULOUSE

The all-time maximum for branching Brownian motion with absorption conditioned on long-time survival

Joint work with Jason Schweinsberg

Pascal Maillard (Université de Toulouse)

Branching processes and applications

May 23, 2023

Overview

1. The model

2. Results

3. Proofs

Branching Brownian motion (BBM)

Definition

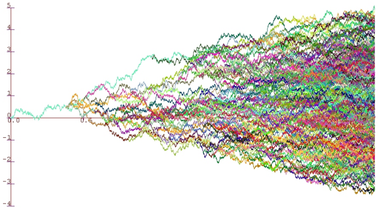
- A particle performs standard Brownian motion started at a point $x \in \mathbb{R}$.
- With rate $1/2$, it branches into 2 offspring (can be generalized)
- Each offspring repeats this process independently of the others.

$R(t)$ = maximum at time t.

Asymptotic speed (Biggins 1977)

Almost surely,

$$R(t)/t \to 1, \quad \text{as } t \to \infty.$$



Picture by Matt Roberts

Branching Brownian motion with absorption

Definition

- Start with one particle at $x \geq 0$.
- Add *drift* -1, to motion of particles.
- Kill particles upon hitting the origin.

Process dies out almost surely Kesten 1978.

Questions

What is the probability of survival until a large time t?
What does the BBM with absorption and critical drift "look like" when conditioned to survive until a large time t?

Context: Genealogy of branching Markov processes

Critical Bienaymé-Galton-Watson process $(Z_n)_{n\geq 0}$, unit variance.

- $\mathbb{P}(\text{survival until generation } n) \sim 2/n$, $n \to \infty$ Kolmogorov 1938
- $(Z_{\lfloor sn \rfloor}/n)_{s\in[0,1]}$ cond. on $Z_n > 0$ converges in law to a cond. Feller diffusion.

Context: Genealogy of branching Markov processes

Critical Bienaymé-Galton-Watson process $(Z_n)_{n\geq 0}$, unit variance.

- $\mathbb{P}(\text{survival until generation } n) \sim 2/n$, $n \to \infty$ Kolmogorov 1938
- $(Z_{\lfloor sn \rfloor}/n)_{s\in[0,1]}$ cond. on $Z_n > 0$ converges in law to a cond. Feller diffusion.

Question

Does this hold for general multi-type branching processes?

Context: Genealogy of branching Markov processes

Critical Bienaymé-Galton-Watson process $(Z_n)_{n\geq 0}$, unit variance.

- $\mathbb{P}(\text{survival until generation } n) \sim 2/n$, $n \to \infty$ Kolmogorov 1938
- $(Z_{\lfloor sn \rfloor}/n)_{s\in[0,1]}$ cond. on $Z_n > 0$ converges in law to a cond. Feller diffusion.

Question

Does this hold for general multi-type branching processes?

Yes for finite number of types. In general (infinite number of types), need a certain integrability condition. Aïdékon, de Raphélis, Harris, Horton, Kyprianou, Palau, Powell, Tourniaire, Wang, . . .

Results

Survival probability

$L_t = ct^{1/3}$, $c = (3\pi^2/2)^{1/3}$, $w_t(x) = L_t \sin(\pi x/L_t)e^{x-L_t}$.

Theorem (M.-Schweinsberg, 2022)

If $x = x(t)$ such that $L_t - x \to \infty$, for some constant $\alpha > 0$,

$$\mathbb{P}_x(\textit{survival until time } t) \sim \alpha w_t(x).$$

- Throughout the article: asymptotics as $t \to \infty$.
- Previous results by Berestycki-Berestycki-Schweinsberg 2012, Kesten 1978, *Derrida-Simon 2007*
- $t^{1/3}$ scaling reminiscent of results about particles in BBM staying always close to the maximum Faraud-Hu-Shi 2012, Fang-Zeitouni 2012, Roberts 2015.

Survival probability (contd.)

$L_t = ct^{1/3}$, $c = (3\pi^2/2)^{1/3}$, $w_t(x) = L_t \sin(\pi x/L_t) e^{x-L_t}$.

Theorem (MS22)

If $x = x(t)$ such that $L_t - x \to \infty$, for some constant $\alpha > 0$,

$$\mathbb{P}_x(\textit{survival until time } t) \sim \alpha w_t(x).$$

ζ = time of extinction.

Corollary (MS22)

Suppose $x = x(t)$ such that $L_t - x \to \infty$. Let $V \sim \mathrm{Exp}(1)$. Then,

$$\frac{\zeta - t}{t^{2/3}} \Longrightarrow \frac{3}{c} V, \quad \textit{under } \mathbb{P}_x(\cdot \,|\, \zeta > t).$$

Yaglom limit

$L_t = ct^{1/3}$, $c = (3\pi^2/2)^{1/3}$, ζ = time of extinction, $R(t) = \max_u X_u(t)$.

Theorem (MS22)

Suppose $x = x(t)$ such that $L_t - x \to \infty$. Let $V \sim \mathrm{Exp}(1)$. Then,

$$\frac{R(t)}{t^{2/9}} \Longrightarrow (3c^2 V)^{1/3}, \quad \textit{under } \mathbb{P}_x(\cdot \,|\, \zeta > t).$$

Reason: morally, $R_t \approx L_{\zeta - t}$ if $\zeta > t$ (and $R_t = 0$ if $\zeta \leq t$). (TODO: picture)

Same result holds with R_t replaced by $\log \#\mathcal{N}_t$.

New result: All-time maximum

Define

$$\mathfrak{M} = \max_{s \geq 0} R(s), \quad \mathfrak{m} = \arg\max_{s \geq 0} R(s).$$

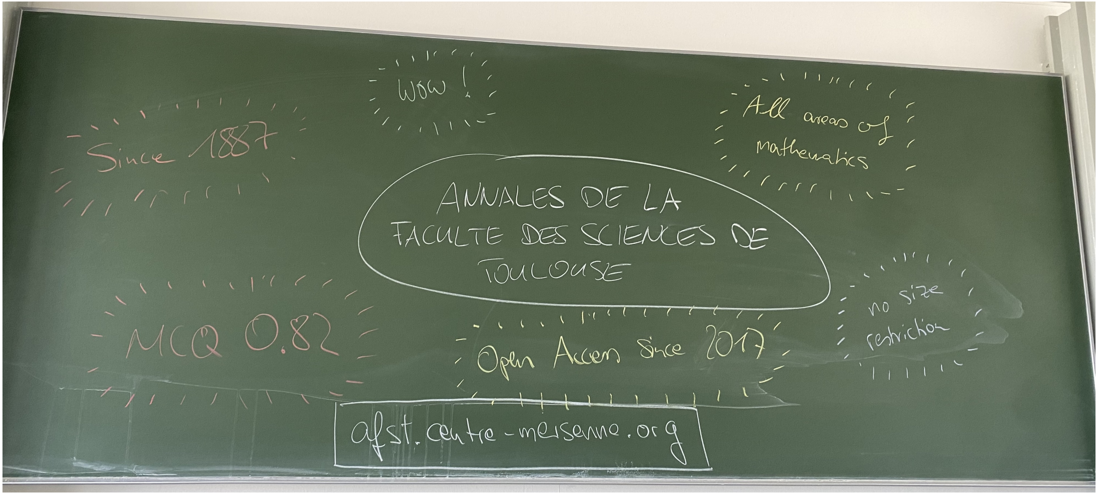
Theorem (M.–Schweinsberg, 2023+)

Suppose $x = x(t)$ such that $L_t - x \gg t^{1/6}$. Then as $t \to \infty$, conditional on $\zeta > t$ we have the convergence in law

$$\left(\frac{L_t - \mathfrak{M}}{t^{1/6}}, \frac{\mathfrak{m}}{t^{5/6}} \right) \Rightarrow \left(c^{1/2} R, 3c^{-1/2} U R \right), \tag{1}$$

where R and U are independent random variables, R is Rayleigh distributed with density $2re^{-r^2}$ on $\mathbb{R}_+$, and U has a uniform distribution on $[0,1]$.

Proofs



https://afst.centre-mersenne.org/

Workhorse: the process $Z_t(s)$

- $L_t(s) = L_{t-s} = c(t-s)^{1/3}$.
- $Z_t(s) = \sum_{u \in \mathbb{N}_s} w_{t-s}(X_u(s)) = \sum_{u \in \mathbb{N}_s} L_t(s) \sin(\pi X_u(s)/L_t(s)) e^{X_u(s) - L_t(s)}$.

Theorem (MS22)

Suppose we start with an initial configuration of particles so that $Z_t(0) \to z_0 > 0$ and $L_t - R(0) \to \infty$. Then there exists a non-degenerate stochastic process $(\Xi(s))_{s \geq 0}$, such that the following convergence in law holds (w.r.t finite-dimensional distributions):

$$(Z_t(t(1-e^{-s})))_{s \geq 0} \Longrightarrow (\Xi(s))_{s \geq 0}, \quad \text{under } \mathbb{P}_x(\cdot \,|\, \zeta > t).$$

In fact, $(\Xi(s))_{s \geq 0}$ is the continuous-state branching process (CSBP) with branching mechanism $\psi(u) = au + \frac{2}{3} u \log u$, for some $a \in \mathbb{R}$, started from z_0.

Workhorse: the process $Z_t(s)$

- $L_t(s) = L_{t-s} = c(t-s)^{1/3}$.
- $Z_t(s) = \sum_{u \in \mathbb{N}_s} w_{t-s}(X_u(s)) = \sum_{u \in \mathbb{N}_s} L_t(s) \sin(\pi X_u(s)/L_t(s)) e^{X_u(s) - L_t(s)}$.
- $(\Xi(s))_{s \geq 0}$: CSBP with branching mechanism $\psi(u) = au + \frac{2}{3} u \log u$, for some $a \in \mathbb{R}$.

Theorem (MS22)

$Z_t(0) \to z_0 > 0$, $L_t - R(0) \to \infty$. Then, $(Z_t(t(1-e^{-s})))_{s \geq 0} \Rightarrow (\Xi(s))_{s \geq 0}$ under $\mathbb{P}_x(\cdot \,|\, \zeta > t)$.

Theorem (MS22)

Under the same assumptions,

$$\mathbb{P}(\zeta > t) \to \mathbb{P}(\Xi(s) \to \infty, \text{ as } s \to \infty \mid \Xi(0) = z_0) = 1 - \exp(-\alpha z_0).$$

Workhorse: the process $Z_t(s)$

- $L_t(s) = L_{t-s} = c(t-s)^{1/3}$.
- $Z_t(s) = \sum_{u \in \mathbb{N}_s} w_{t-s}(X_u(s)) = \sum_{u \in \mathbb{N}_s} L_t(s) \sin(\pi X_u(s)/L_t(s)) e^{X_u(s) - L_t(s)}$.
- $(\Xi(s))_{s \geq 0}$: CSBP with branching mechanism $\psi(u) = au + \frac{2}{3} u \log u$, for some $a \in \mathbb{R}$.

Theorem (MS22)

Suppose $x = x(t)$ such that $L_t - x \to \infty$. Then there exists a non-degenerate stochastic process $(\Phi(s))_{s \geq 0}$, such that the following convergence in law holds (w.r.t finite-dimensional distributions):

$$(Z_t(t(1-e^{-s})))_{s \geq 0} \Longrightarrow (\Phi(s))_{s \geq 0}, \quad \text{under } \mathbb{P}_x(\cdot \,|\, \zeta > t).$$

$(\Phi(s))_{s \geq 0}$ is the process $(\Xi(s))_{s \geq 0}$, started from 0 and conditioned to go to ∞, i.e. the descendance of a "prolific individual" Bertoin–Fontbona–Martínez 2008

An auxiliary process with spine

Branching Brownian motion with a distinguished particle (the *spine*):

- Starts at time zero from a single particle, the spine, at a point $x \in [0, L_t]$.
- For $s \in [0, t]$, define
$$\tau(s) = \int_0^s \frac{1}{L_t(u)^2}\,du.$$
The spine's trajectory is equal in law to the process $(L_t(s)K(\tau(s)))_{s\in[0,t]}$, where $(K(u))_{u\geq 0}$ is the *Brownian taboo process* on $[0, 1]$ (next slide) started at $x/L_t(0)$.
- The spine branches at the accelerated rate $(m+1)\beta$ and according to the size-biased offspring distribution.
- One of these offspring is chosen randomly to continue as the spine. The others spawn independent copies of the original process started at their positions.

Brownian taboo process Knight 1969

"Brownian motion conditioned to stay in the interval $[0, 1]$"

Doob h-transform of Brownian motion killed when it exists the interval $[0, 1]$ w.r.t. the $-(\pi^2/2)$-harmonic function $h(x) = \sin(\pi x)$

Infinitesimal generator: $\frac{1}{2}\partial_x^2 + \pi \cot(\pi x)\partial_x$

Boundary $\{0, 1\}$ is *entrance-not-exit*

Small time: conditioning = spine

Lemma

Suppose $x = x(t)$ such that $L_t - x \to \infty$. Let μ_s be the law of the original process conditioned on $\{\zeta > t\}$, and let ν_s be the law of the process with spine, both starting with one particle at x. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that for sufficiently large t, we have

$$d_{TV}(\mu_{\delta t}, \nu_{\delta t}) \leq \varepsilon,$$

where d_{TV} is the total variation distance between measures.

Proof uses a measure change w.r.t. $Z_t(\delta t)$. Recall: $\mathbb{P}(\zeta > t \mid \mathcal{F}_{\delta t}) \approx \alpha Z_t(\delta t)$.

Proof of theorem

- $\mathfrak{M} = \max_{s\geq 0} R(s)$, $\mathfrak{m} = \arg\max_{s\geq 0} R(s)$
- $x = x(t)$ such that $L_t - x \gg t^{1/6} \asymp L_t^{1/2}$
- To show: $\left(\frac{L_t - \mathfrak{M}}{L_t^{1/2}}, L_t^{1/2}\frac{\mathfrak{m}}{t}\right) \Rightarrow (R, 3UR)$, conditioned on $\zeta > t$. R Rayleigh, U uniform.

Proof steps

1. Conditioning $\to$ auxiliary process with spine
2. Maximum, argmax $\approx$ maximum, argmax of spine
3. Use excursion theory for Brownian taboo process

Step 3

Recall: spine is $L_t(s)(1-K(\tau(s)))$, $\tau(s)=\int_0^s \frac{1}{L_t(u)^2}\,du$
$\mathfrak{M}=\max_{s\geq 0} L_t(s)(1-K(\tau(s)))$, $\mathfrak{m}=\arg\max_{s\geq 0} L_t(s)(1-K(\tau(s)))$

Linearization

$$\left(\frac{L_t-\mathfrak{M}}{L_t^{1/2}}, L_t^{1/2}\frac{\mathfrak{m}}{t}\right)\approx\sqrt{\frac{\pi^2}{2}}\left(\frac{1}{\sqrt{\gamma}}\min_{s\geq 0}\{K(s)+\gamma s\}, 3\sqrt{\gamma}\,\underset{s\geq 0}{\operatorname{argmin}}\{K(s)+\gamma s\}\right), \gamma=\frac{\pi^2}{2L_t}$$

Step 3

Recall: spine is $L_t(s)(1-K(\tau(s)))$, $\tau(s) = \int_0^s \frac{1}{L_t(u)^2}\,du$
$\mathfrak{M} = \max_{s\geq 0} L_t(s)(1-K(\tau(s)))$, $\mathfrak{m} = \arg\max_{s\geq 0} L_t(s)(1-K(\tau(s)))$

Linearization

$$\left(\frac{L_t-\mathfrak{M}}{L_t^{1/2}}, L_t^{1/2}\frac{\mathfrak{m}}{t}\right) \approx \sqrt{\frac{\pi^2}{2}}\left(\frac{1}{\sqrt{\gamma}}\min_{s\geq 0}\{K(s)+\gamma s\}, 3\sqrt{\gamma}\,\underset{s\geq 0}{\operatorname{argmin}}\{K(s)+\gamma s\}\right), \gamma = \frac{\pi^2}{2L_t}$$

Decompose into excursions below 1/2

$$\left(\frac{1}{\sqrt{\gamma}}\min_{s\geq 0}\{K(s)+\gamma s\}, \sqrt{\gamma}\,\underset{s\geq 0}{\operatorname{argmin}}\{K(s)+\gamma s\}\right) \approx \left(\frac{1}{\sqrt{\gamma}}\min_{(s,H_s)\in\Pi}(H_s+\gamma s), \sqrt{\gamma}\,\underset{(s,H_s)\in\Pi}{\operatorname{argmin}}(H_s+\gamma s)\right),$$

where Π is a Poisson point process on $\mathbb{R}_+ \times [0,1/2]$ with intensity measure $du \otimes \nu$, with $\nu(0,h) = \pi\tan(\pi h)$ Lambert 2000.

A Poisson process calculation

Change variables $u = \sqrt{\gamma} s$, $H_u = H_s/\sqrt{\gamma}$.

$$\left(\frac{1}{\sqrt{\gamma}} \min_{(s,H_s)\in\Pi} (H_s + \gamma s), \sqrt{\gamma} \operatorname{argmin}_{(s,H_s)\in\Pi} (H_s + \gamma s) \right) = \left(\min_{(u,H_u)\in\Pi_\gamma} (H_u + u), \operatorname{argmin}_{(u,H_u)\in\Pi_\gamma} (H_u + u) \right)$$

where Π_γ is a Poisson point process with intensity measure $du/\sqrt{\gamma} \otimes (1/\sqrt{\gamma})_* \nu$.

$$\Pi_\gamma \xrightarrow{\gamma \to 0} \mathsf{PPP}(du \otimes \pi^2 dh)$$

$$\left(\min_{(u,H_u)\in\Pi} H_u, \operatorname{argmin}_{(u,H_u)\in\Pi} H_u \right) \sim \sqrt{\frac{2}{\pi^2}} (R, UR)$$

Thank you for your attention!