Diffusion Approximation of Controlled Multi-type Branching Processes

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1. Introduction

Multi-type branching processes can be well-applied to describe evolutionary systems in which different types of elements coexist. We focus on the class of Controlled Multi-type Branching Processes (CMBPs) whose key feature is that the number of parents of each type at a given generation is determined by a random control mechanism that depends on the number of individuals of different types in the previous generation.

Aim

We study the asymptotic behaviour of CMBPs. A Feller type diffusion approximation for some CMBPs is derived, obtaining a natural extension of the single-type functional limit theorem by González et al. (2023) to the multi-type case. The results presented here are gathered in Barczy et al. (2023).

2. Probability Model

Definition

For a fixed $p \in \mathbb{N}$, let us consider a Controlled p–type Branching Process $(\boldsymbol{Z}_k)_{k\in\mathbb{Z}_+}$, defined recursively as

$$\boldsymbol{Z}_{k+1} := \sum_{i=1}^{p} \sum_{j=1}^{\phi_{k,i}(\boldsymbol{Z}_k)} \boldsymbol{X}_{k,j,i}, \qquad k \in \mathbb{Z}_+, \tag{1}$$

where $\boldsymbol{Z}_k =: (Z_{k,1}, \ldots, Z_{k,p})^\top$, $\boldsymbol{\phi}_k(\boldsymbol{z}) =: (\phi_{k,1}(\boldsymbol{z}), \ldots, \phi_{k,p}(\boldsymbol{z}))^\top$, with $\boldsymbol{z} \in \mathbb{Z}_+^p$, and $\boldsymbol{X}_{k,j,i} =: (X_{k,j,i,1}, \ldots, X_{k,j,i,p})^\top$ are $\mathbb{Z}_+^p$–valued random vectors.

Assume that $\{\boldsymbol{Z}_0, \boldsymbol{\phi}_k(\boldsymbol{z}), \boldsymbol{X}_{k,j,i} : k \in \mathbb{Z}_+,\ j \in \mathbb{N},\ \boldsymbol{z} \in \mathbb{Z}_+^p,\ i \in \{1,\ldots,p\}\}$ are independent, the control distributions $\{\boldsymbol{\phi}_k(\boldsymbol{z}) : k \in \mathbb{Z}_+\}$ are identically distributed for each $\boldsymbol{z} \in \mathbb{Z}_+^p$ and the offspring distributions $\{\boldsymbol{X}_{k,j,i} : k \in \mathbb{Z}_+,\ j \in \mathbb{N}\}$ are also identically distributed for each $i \in \{1,\ldots,p\}$.

Intuitive Interpretation

- $Z_{k,i}$ is the number of i–type individuals in the k–th generation.
- $\phi_{k,i}(\boldsymbol{Z}_k)$ is the number of i–type progenitors in the k–th generation.
- $X_{k,j,i,l}$ is the number of l–type offsprings of the j–th i–type progenitor in the k–th generation.

Some Particular Cases of Controlled Multi-type Branching Processes

- Multi-type Branching Process with Immigration (MBPI), $(\boldsymbol{Y}_k)_{k\in\mathbb{Z}_+}$, given by

$$\boldsymbol{Y}_{k+1} := \sum_{i=1}^{p} \sum_{j=1}^{Y_{k,i}} \boldsymbol{\xi}_{k,j,i} + \boldsymbol{I}_{k+1}, \qquad k \in \mathbb{Z}_+, \tag{2}$$

  where $\{\boldsymbol{Y}_0, \boldsymbol{\xi}_{k,j,i}, \boldsymbol{I}_k : k \in \mathbb{Z}_+,\ j \in \mathbb{N},\ i \in \{1,\ldots,p\}\}$ are independent $\mathbb{Z}_+^p$–valued random vectors, $\{\boldsymbol{\xi}_{k,j,i} : k \in \mathbb{Z}_+,\ j \in \mathbb{N}\}$ are identically distributed for each $i \in \{1,\ldots,p\}$ (offspring distributions) and $\{\boldsymbol{I}_k : k \in \mathbb{Z}_+\}$ are also identically distributed (immigration distribution).

- 2–Sex Branching Process with Immigration (2SBPI), $(F_k, M_k)_{k\in\mathbb{Z}_+}$, given by

$$\begin{aligned} U_k &:= L(F_k, M_k), \qquad k \in \mathbb{Z}_+, \\ (F_{k+1}, M_{k+1}) &:= \sum_{j=1}^{U_k} (f_{k,j}, m_{k,j}) + (F_{k+1}^I, M_{k+1}^I), \qquad k \in \mathbb{Z}_+, \end{aligned} \tag{3}$$

  where (F_0, M_0) is the random initial generation, $\{(F_0, M_0), (f_{k,j}, m_{k,j}), (F_k^I, M_k^I) : k \in \mathbb{Z}_+,\ j \in \mathbb{N}\}$ are independent $\mathbb{Z}_+^2$–valued random vectors, $\{(f_{k,j}, m_{k,j}) : k \in \mathbb{Z}_+,\ j \in \mathbb{N}\}$ are identically distributed (offspring distribution), and $\{(F_k^I, M_k^I) : k \in \mathbb{Z}_+\}$ are also identically distributed (immigration distribution). Further, $(U_k)_{k\in\mathbb{Z}_+}$ is a sequence of mating units corresponding to the mating function $L : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{Z}_+$.

Notation and Assumptions

We suppose that $\mathrm{E}\big[\|\boldsymbol{X}_{0,1,i}\|^4\big] < \infty$ for $i = 1,\ldots,p$ and $\mathrm{E}\big[\|\boldsymbol{\phi}_0(\boldsymbol{z})\|^4\big] < \infty$ for $\boldsymbol{z} \in \mathbb{Z}_+^p$, and we denote $\boldsymbol{m}_i := \mathrm{E}\big[\boldsymbol{X}_{0,1,i}\big] \in \mathbb{R}_+^p$, $\boldsymbol{\varepsilon}(\boldsymbol{z}) := \mathrm{E}[\boldsymbol{\phi}_0(\boldsymbol{z})] \in \mathbb{R}_+^p$, $\Sigma_i := \mathrm{Var}\big[\boldsymbol{X}_{0,1,i}\big] \in \mathbb{R}^{p\times p}$, $\Gamma(\boldsymbol{z}) := \mathrm{Var}[\boldsymbol{\phi}_0(\boldsymbol{z})] \in \mathbb{R}^{p\times p}$ and $\kappa_i(\boldsymbol{z}) := \mathrm{E}\big[(\phi_{0,i}(\boldsymbol{z}) - \varepsilon_i(\boldsymbol{z}))^4\big] \in \mathbb{R}_+$, where $i, l \in \{1,\ldots,p\}$ and $\boldsymbol{z} \in \mathbb{Z}_+^p$.

Let us consider the matrix $\mathsf{m} := (\boldsymbol{m}_1, \ldots, \boldsymbol{m}_p) \in \mathbb{R}_+^{p\times p}$.

Hypothesis 1. $\mathrm{E}\big[\|\boldsymbol{Z}_0\|^2\big]$, $\mathrm{E}\big[\|\boldsymbol{X}_{0,1,i}\|^4\big]$ and $\mathrm{E}\big[\|\boldsymbol{\phi}_0(\boldsymbol{z})\|^4\big]$ are finite for each $i \in \{1,\ldots,p\}$ and $\boldsymbol{z} \in \mathbb{Z}_+^p$.

Hypothesis 2. There exist $\Lambda \in \mathbb{R}^{p\times p}$, $\boldsymbol{\alpha} \in \mathbb{R}^p$ and a function $\boldsymbol{g} : \mathbb{Z}_+^p \to \mathbb{R}^p$ with $\|\boldsymbol{g}(\boldsymbol{z})\| = \mathrm{o}(1)$ as $\|\boldsymbol{z}\| \to \infty$ such that $\boldsymbol{\varepsilon}(\boldsymbol{z}) = \Lambda\boldsymbol{z} + \boldsymbol{\alpha} + \boldsymbol{g}(\boldsymbol{z})$, $\boldsymbol{z} \in \mathbb{Z}_+^p$.

Hypothesis 3. $\|\Gamma(\boldsymbol{z})\| = \mathrm{o}(\|\boldsymbol{z}\|)$ as $\|\boldsymbol{z}\| \to \infty$.

Hypothesis 4. $\kappa_i(\boldsymbol{z}) = \mathrm{O}(\|\boldsymbol{z}\|^2)$ as $\|\boldsymbol{z}\| \to \infty$ for $i = 1,\ldots,p$.

Hypothesis 5. The matrix $\tilde{\mathsf{m}} := \mathsf{m}\Lambda$ belongs to $\mathbb{R}_+^{p\times p}$, $\tilde{\rho} := 1$ is an eigenvalue of $\tilde{\mathsf{m}}$ having algebraic and geometric multiplicities 1, and the absolute values of the other eigenvalues of $\tilde{\mathsf{m}}$ are less than 1. There exist a unique right eigenvector $\tilde{\boldsymbol{u}} \in \mathbb{R}_+^p$ and a unique left eigenvector $\tilde{\boldsymbol{v}} \in \mathbb{R}_+^p$ corresponding to $\tilde{\rho} = 1$ such that $\tilde{u}_1 + \ldots + \tilde{u}_p = 1$, $\Lambda\tilde{\boldsymbol{u}} \in \mathbb{R}_+^p$ and $\tilde{\boldsymbol{v}}^\top\tilde{\boldsymbol{u}} = 1$. Finally, $\lim_{k\to\infty}\tilde{\mathsf{m}}^k = \tilde{\Pi}$ and there exist $\tilde{c} \in \mathbb{R}_{++}$ and $\tilde{r} \in (0,1)$ such that $\|\tilde{\mathsf{m}}^k - \tilde{\Pi}\| \le \tilde{c}\tilde{r}^k$ for each $k \in \mathbb{N}$, where $\tilde{\Pi} := \tilde{\boldsymbol{u}}\tilde{\boldsymbol{v}}^\top$.

Hypothesis 6. For all $\epsilon > 0$ and $B > 0$, there exists $k_0(\epsilon, B) \in \mathbb{N}$ such that $\mathrm{P}[\|\boldsymbol{Z}_k\| \le B] \le \epsilon$ for each $k \ge k_0(\epsilon, B)$, $k \in \mathbb{N}$.

Remark.

- If the matrix $\tilde{\mathsf{m}}$ is primitive, its Perron–Frobenius eigenvalue $\tilde{\rho}$ equals 1, and $\Lambda\tilde{\boldsymbol{u}} \in \mathbb{R}_+^p$, where $\tilde{\boldsymbol{u}}$ is the right Perron–Frobenius eigenvector of $\tilde{\mathsf{m}}$ corresponding to $\tilde{\rho}$, then Hypothesis 5 holds.
- A sufficient condition for Hypothesis 6 is the almost sure explosion of the process, i.e. $\mathrm{P}[\|\boldsymbol{Z}_k\| \to \infty \text{ as } k \to \infty] = 1$.
- In case of $\boldsymbol{g} \equiv \boldsymbol{0}_p$, the Hypothesis 6 is not necessary.

3. Results

Let $\mathbf{D}(\mathbb{R}_+, \mathbb{R}^p)$ be the space of $\mathbb{R}^p$–valued càdlàg functions on $\mathbb{R}_+$ with the Skorokhod metric and $\xrightarrow{\mathcal{L}}$ the weak convergence of distributions of stochastic processes on the space. Let us denote the convergence in probability by $\xrightarrow{\mathrm{P}}$ and consider the operator $\odot : \mathbb{R}^p \times (\mathbb{R}^{p\times p})^p \to \mathbb{R}^{p\times p}$, $\boldsymbol{z} \odot \boldsymbol{\Sigma} := \sum_{i=1}^p z_i\Sigma_i$ for $\boldsymbol{z} = (z_1, \ldots, z_p)^\top \in \mathbb{R}^p$ and $\boldsymbol{\Sigma} = (\Sigma_1, \ldots, \Sigma_p) \in (\mathbb{R}^{p\times p})^p$.

Main Scaling Limit Result

Theorem. *Suppose that Hypotheses 1–6 hold for the CMBP $(\boldsymbol{Z}_k)_{k\in\mathbb{Z}_+}$ given in (1). Then*

$$(n^{-1}\boldsymbol{Z}_{\lfloor nt\rfloor})_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{L}} (\mathcal{Z}_t\tilde{\boldsymbol{u}})_{t\in\mathbb{R}_+} \qquad \text{as } n \to \infty,$$

where $(\mathcal{Z}_t)_{t\in\mathbb{R}_+}$ is the pathwise unique strong solution of the SDE

$$\mathrm{d}\mathcal{Z}_t = \tilde{\boldsymbol{v}}^\top\mathsf{m}\boldsymbol{\alpha}\,\mathrm{d}t + \sqrt{\tilde{\boldsymbol{v}}^\top((\Lambda\tilde{\boldsymbol{u}}) \odot \boldsymbol{\Sigma})\tilde{\boldsymbol{v}}\mathcal{Z}_t^+}\,\mathrm{d}\mathcal{W}_t, \qquad t \in \mathbb{R}_+,$$

with initial value $\mathcal{Z}_0 = 0$, where $(\mathcal{W}_t)_{t\in\mathbb{R}_+}$ is a standard Wiener process.

Applications

Diffusion Approximation of Multi-type Branching Processes with Immigration

Corollary 1 [Ispány and Pap (2014)]. *Let $(\boldsymbol{Y}_k)_{k\in\mathbb{Z}_+}$ be a critical primitive p–type branching process with immigration, i.e. a MBPI defined in (2) such that the offspring mean matrix $\mathsf{m}_{\boldsymbol{\xi}} := (\mathrm{E}\big[\boldsymbol{\xi}_{0,1,1}\big], \ldots, \mathrm{E}\big[\boldsymbol{\xi}_{0,1,p}\big]) \in \mathbb{R}_+^{p\times p}$ is primitive with Perron–Frobenius eigenvalue 1. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be, respectively, its right and left Perron–Frobenius eigenvectors, $\boldsymbol{m}_{\boldsymbol{I}} := \mathrm{E}[\boldsymbol{I}_1] \in \mathbb{R}_+^p$ and $\mathbf{V} := (\mathrm{Var}\big[\boldsymbol{\xi}_{0,1,1}\big], \ldots, \mathrm{Var}\big[\boldsymbol{\xi}_{0,1,p}\big]) \in (\mathbb{R}^{p\times p})^p$. Assume $\mathrm{E}\big[\|\boldsymbol{Y}_0\|^2\big] < \infty$, $\mathrm{E}\big[\|\boldsymbol{\xi}_{0,1,i}\|^4\big] < \infty$, $i \in \{1,\ldots,p\}$, and $\mathrm{E}\big[\|\boldsymbol{I}_1\|^4\big] < \infty$. Then*

$$(n^{-1}\boldsymbol{Y}_{\lfloor nt\rfloor})_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{L}} (\mathcal{Y}_t\boldsymbol{u})_{t\in\mathbb{R}_+} \qquad \text{as } n \to \infty,$$

where $(\mathcal{Y}_t)_{t\in\mathbb{R}_+}$ is the pathwise unique strong solution of the SDE

$$\mathrm{d}\mathcal{Y}_t = \boldsymbol{v}^\top\boldsymbol{m}_{\boldsymbol{I}}\,\mathrm{d}t + \sqrt{\boldsymbol{v}^\top(\boldsymbol{u} \odot \mathbf{V})\boldsymbol{v}\mathcal{Y}_t^+}\,\mathrm{d}\mathcal{W}_t, \qquad t \in \mathbb{R}_+,$$

with initial value $\mathcal{Y}_0 = 0$, where $(\mathcal{W}_t)_{t\in\mathbb{R}_+}$ is a standard Wiener process.

Diffusion Approximation of 2–Sex Branching Processes with Immigration

Corollary 2. *Let $(F_k, M_k)_{k\in\mathbb{Z}_+}$ be a 2SBPI defined in (3) with the promiscuous mating function, $L(x,y) := x\min\{1,y\}$, $x, y \in \mathbb{Z}_+$. Assume that (F_0, M_0) is a $\mathbb{N}^2$–valued random vector, $\mathrm{E}\big[\|(F_0, M_0)\|^2\big] < \infty$, $\mathrm{E}\big[\|(f_{0,1}, m_{0,1})\|^4\big] < \infty$, and $\mathrm{E}\Big[\|(F_1^I, M_1^I)\|^4\Big] < \infty$. If $\mathrm{P}\Big[M_1^I = 0\Big] = 0$ and $\mathrm{E}\big[f_{0,1}\big] = 1$, then*

$$(n^{-1}(F_{\lfloor nt\rfloor}, M_{\lfloor nt\rfloor}))_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{L}} (\mathcal{X}_t(1, \mathrm{E}\big[m_{0,1}\big]))_{t\in\mathbb{R}_+} \qquad \text{as } n \to \infty,$$

where $(\mathcal{X}_t)_{t\in\mathbb{R}_+}$ is the pathwise unique strong solution of the SDE

$$\mathrm{d}\mathcal{X}_t = \mathrm{E}\Big[F_1^I\Big]\,\mathrm{d}t + \sqrt{\mathrm{Var}\big[f_{0,1}\big]\,\mathcal{X}_t^+}\,\mathrm{d}\mathcal{W}_t, \qquad t \in \mathbb{R}_+,$$

with initial value $\mathcal{X}_0 = 0$, where $(\mathcal{W}_t)_{t\in\mathbb{R}_+}$ is a standard Wiener process.

Corollary 3. *Let $(F_k, M_k)_{k\in\mathbb{Z}_+}$ be a 2SBPI defined in (3) with the self-fertilization mating function, $L(\boldsymbol{z}) := z_1 + z_2$, $\boldsymbol{z} = (z_1, z_2) \in \mathbb{R}_+^2$. Assume that $\mathrm{E}\big[\|(F_0, M_0)\|^2\big] < \infty$, $\mathrm{E}\big[\|(f_{0,1}, m_{0,1})\|^4\big] < \infty$, $\mathrm{E}\Big[\|(F_1^I, M_1^I)\|^4\Big] < \infty$, and $\mathrm{E}\big[f_{0,1}\big], \mathrm{E}\big[m_{0,1}\big] \in (0,1)$ are such that $\mathrm{E}\big[f_{0,1}\big] + \mathrm{E}\big[m_{0,1}\big] = 1$. Then*

$$(n^{-1}(F_{\lfloor nt\rfloor}, M_{\lfloor nt\rfloor}))_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{L}} (\mathcal{X}_t(\mathrm{E}\big[f_{0,1}\big], \mathrm{E}\big[m_{0,1}\big]))_{t\in\mathbb{R}_+} \qquad \text{as } n \to \infty,$$

where $(\mathcal{X}_t)_{t\in\mathbb{R}_+}$ is the pathwise unique strong solution of the SDE

$$\mathrm{d}\mathcal{X}_t = \Big(\mathrm{E}\Big[F_1^I\Big] + \mathrm{E}\Big[M_1^I\Big]\Big)\,\mathrm{d}t + \sqrt{\mathrm{Var}\big[f_{0,1} + m_{0,1}\big]\,\mathcal{X}_t^+}\,\mathrm{d}\mathcal{W}_t, \qquad t \in \mathbb{R}_+,$$

with initial value $\mathcal{X}_0 = 0$, where $(\mathcal{W}_t)_{t\in\mathbb{R}_+}$ is a standard Wiener process.

Limit Distribution for the Relative Frequencies of distinct types of individuals

Corollary 4. *Suppose that Hypotheses 1–4 and 6 hold for the CMBP $(\boldsymbol{Z}_k)_{k\in\mathbb{Z}_+}$ given in (1). Assume also that the matrix $\tilde{\mathsf{m}}$ is primitive with Perron–Frobenius eigenvalue 1, and $\Lambda\tilde{\boldsymbol{u}} \in \mathbb{R}_+^p$. If, in addition, $\tilde{\boldsymbol{v}}^\top\mathsf{m}\boldsymbol{\alpha} > 0$, then for all $t > 0$ and $i, j \in \{1,\ldots,p\}$, we get*

$$\mathbb{1}_{\{\boldsymbol{e}_j^\top\boldsymbol{Z}_{\lfloor nt\rfloor}\neq 0\}}\frac{\boldsymbol{e}_i^\top\boldsymbol{Z}_{\lfloor nt\rfloor}}{\boldsymbol{e}_j^\top\boldsymbol{Z}_{\lfloor nt\rfloor}} \xrightarrow{\mathrm{P}} \frac{\boldsymbol{e}_i^\top\tilde{\boldsymbol{u}}}{\boldsymbol{e}_j^\top\tilde{\boldsymbol{u}}} \quad \text{and} \quad \mathbb{1}_{\{\boldsymbol{Z}_{\lfloor nt\rfloor}\neq\boldsymbol{0}_p\}}\frac{\boldsymbol{e}_i^\top\boldsymbol{Z}_{\lfloor nt\rfloor}}{\sum_{k=1}^p\boldsymbol{e}_k^\top\boldsymbol{Z}_{\lfloor nt\rfloor}} \xrightarrow{\mathrm{P}} \boldsymbol{e}_i^\top\tilde{\boldsymbol{u}} \qquad \text{as } n \to \infty.$$

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