Old and new results on local limits of conditioned trees

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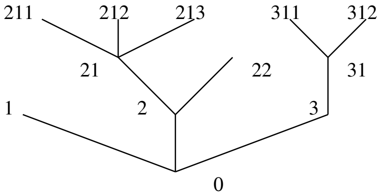
Outline

Notations for discrete trees



Set of nodes $\mathcal{U} = \bigcup_{n\in\mathbb{N}} (\mathbb{N}^*)^n$. For a node $u \in \mathbf{t}$, we set

- number of children : $k_u(\mathbf{t})$ ($< \infty$ for the moment)
- height : $h(u)$

For the tree **t**:

- height : $H(\mathbf{t}) = \max\{h(u), u \in \mathbf{t}\}$
- number of vertices=size of **t** : $|\mathbf{t}|$
- size of the n^{th} generation : $Z_n(\mathbf{t})$

The local topology

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$$\begin{aligned} \mathbf{t}_n \longrightarrow \mathbf{t} &\iff \forall h > 0,\ r_h(\mathbf{t}_n) = r_h(\mathbf{t}) \quad \text{for } n \text{ large enough} \\ &\iff \forall u \in \mathcal{U},\ k_u(\mathbf{t}_n) \longrightarrow k_u(\mathbf{t}) \end{aligned}$$

with the convention $k_u(\mathbf{t}) = -1$ if $u \notin \mathbf{t}$.

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Conditioning on non-extinction

Theorem (Kesten, 1986)

Let p be a critical or sub-critical offspring distribution (mean $\mu \leq 1$).
Let τ_n be a GW(p)-tree conditioned on $H(\tau_n) = n$.
Then

$$\tau_n \xrightarrow{(d)} \tau^*.$$

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Definition of τ^* as a size-biased GW-tree:

$$\forall h > 0, \forall \phi,\ \mathbb{E}\left[\phi(r_h(\tau^*))\right] = \mathbb{E}\left[\frac{Z_h}{\mu^h}\phi(r_h(\tau))\right].$$

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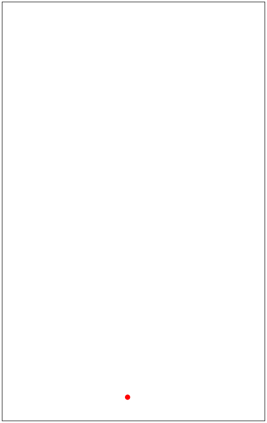

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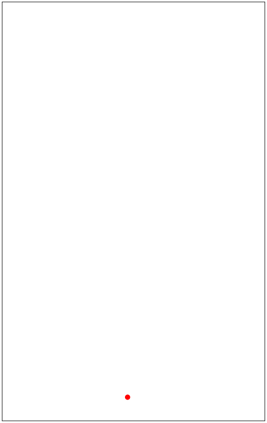

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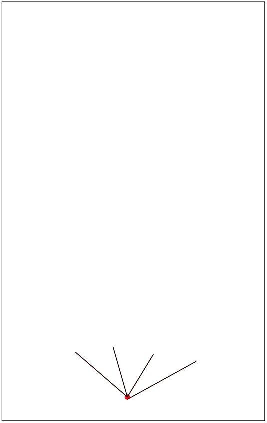

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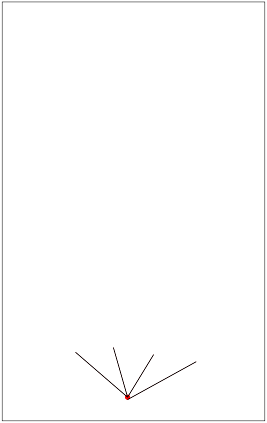

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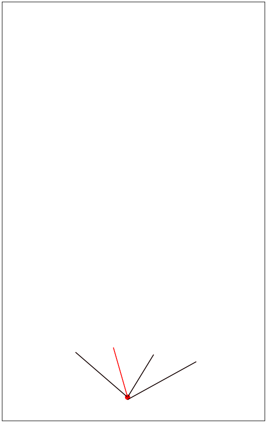

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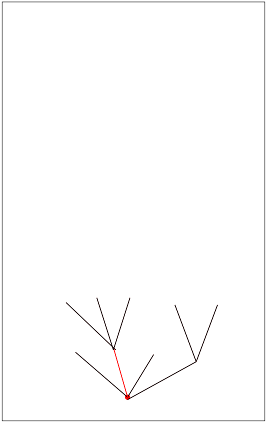

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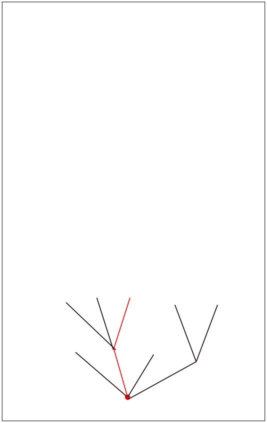

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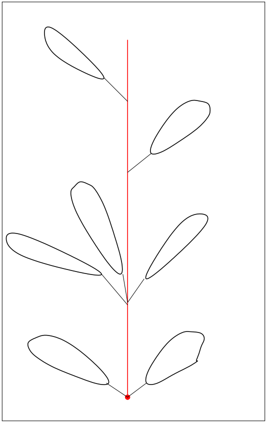

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- Given the $Y_u = n-1$, the number of trees grafted on the left is uniformly distributed on $\{0, \ldots, n-1\}$.

General functionals

Notation: $\mathbf{t}$, $\tilde{\mathbf{t}}$ two trees, x a leaf of $\mathbf{t}$. $\mathbf{t} \circledast_x \tilde{\mathbf{t}}$ is the tree obtained by grafting the tree $\tilde{\mathbf{t}}$ on $\mathbf{t}$ at x.

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Let A be an integer-valued functional on discrete trees.
Assumption : A is additive if there exists a function $D(\mathbf{t}, x) \geq 0$ such that

$$A(\mathbf{t} \circledast_x \tilde{\mathbf{t}}) = A(\tilde{\mathbf{t}}) + D(\mathbf{t}, x)$$

for $A(\tilde{\mathbf{t}})$ large enough.

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Theorem (A.-Delmas, 2014)

Let p be a **critical** *offspring distribution. If*

$$\lim_{n \to +\infty} \frac{\mathbb{P}(A(\tau) = n+1)}{\mathbb{P}(A(\tau) = n)} = 1.$$

Then

$$\mathrm{dist}(\tau | A(\tau) = n) \longrightarrow \mathrm{dist}(\tau^*)$$

Examples

- Height of the tree :$H(\mathbf{t} \circledast_x \tilde{\mathbf{t}}) = H(\tilde{\mathbf{t}}) + h(x)$.

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- Total progeny: $|\mathbf{t} \circledast_x \tilde{\mathbf{t}}| = |\tilde{\mathbf{t}}| + |\mathbf{t}| - 1$.
- Number of leaves.
- Number of nodes with given out-degree. $\mathcal{A} \subset \mathbb{N}$.
  $L_{\mathcal{A}}(\mathbf{t}) = Card\{u \in \mathbf{t},\ k_u(\mathbf{t}) \in \mathcal{A}\}$.

Outline

Very large geometric trees

Goal : $\lim_{n\to+\infty} dist(\tau|Z_n = a_n)$.

Offspring distribution :

$$\begin{cases} p(0) = 1-\eta \\ p(k) = \eta q(1-q)^{k-1} \quad \text{for } k \geq 1. \end{cases}$$

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$$\text{Set } c_n = \begin{cases} \mu^{-n} & \text{if } \mu = \eta/q < 1, \\ n^2 & \text{if } \mu = 1, \\ \mu^n & \text{if } \mu > 1. \end{cases}$$

Result

Theorem (A.-Bouaziz-Delmas, 2020)

Suppose that $a_n > 0$ *and* $\lim_{n\to+\infty} \frac{a_n}{c_n} = \theta \in [0, +\infty]$
Then, the distribution of τ *conditionally given* $\{Z_n = a_n\}$ *converges to the distribution of some random tree* τ^θ*.*

Description of τ^0 and τ^∞

- τ^0 is the Kesten tree associated with $\tilde{p}(n) := \kappa_e^n p(n)$ where $\kappa_e = \min\left(\frac{1-\eta}{1-q}, 1\right)$ is the extinction probability.

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- The tree τ^∞:
  - Its root has infinite degree.
  - The sub-trees are i.i.d inhomogeneous GW trees with offspring distribution at height h given by

$$p_h^\infty(k) = \frac{\gamma_{h+1}^k}{\gamma_h} p(k),$$

where,

$$\gamma_h = \begin{cases} \frac{\kappa-\mu^h}{1-\mu^h} & \text{if } \mu \neq 1 \\ 1 + \frac{q}{h(1-q)} & \text{if } \mu = 1 \end{cases}$$

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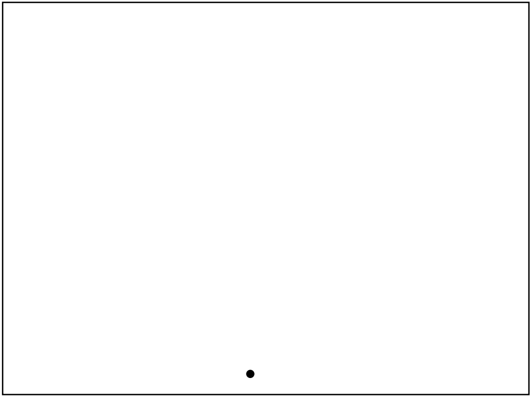
We have

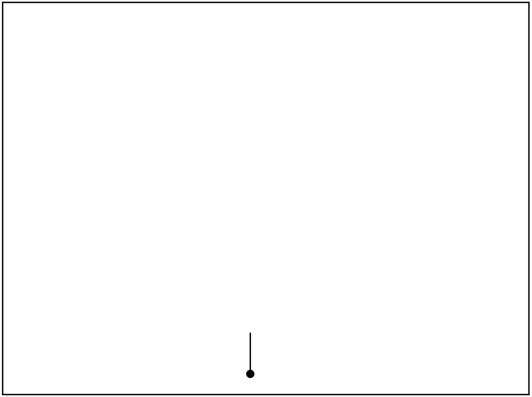
$$\mu_h := \sum_{k=1}^{\infty} k p_h^\infty(k) > 1 \quad \text{and} \quad \lim_{h\to+\infty} \mu_h = \mu^{\pm 1}.$$

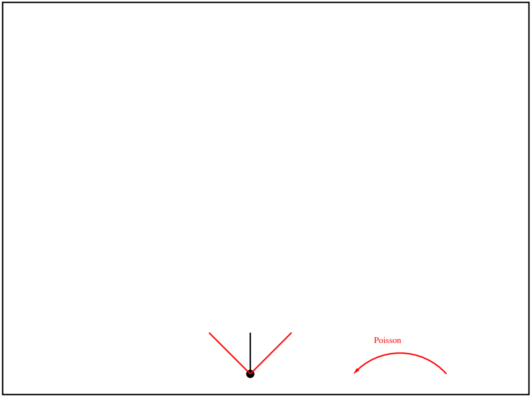
The skeleton of τ^θ

Suppose we have k individuals at height h.

- Each individual gives birth to a single individual at height $h+1$.
- A $Poisson(\theta\zeta_h)$ number of supplementary individuals appear and are attached uniformly on the k initial individuals.
- $$\zeta_h = \begin{cases} cste \cdot \mu^{-h} & \text{if } \mu < 1, \\ cste & \text{if } \mu = 1. \\ cste \cdot \mu^h & \text{if } \mu > 1. \end{cases}$$



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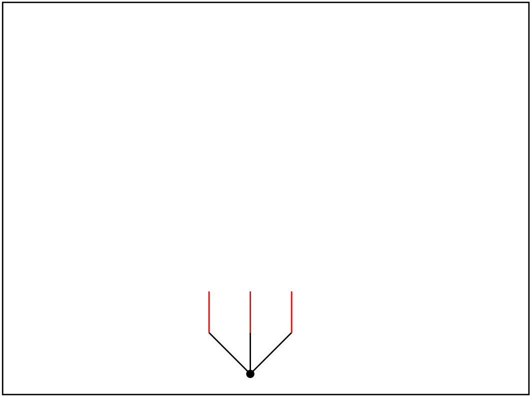

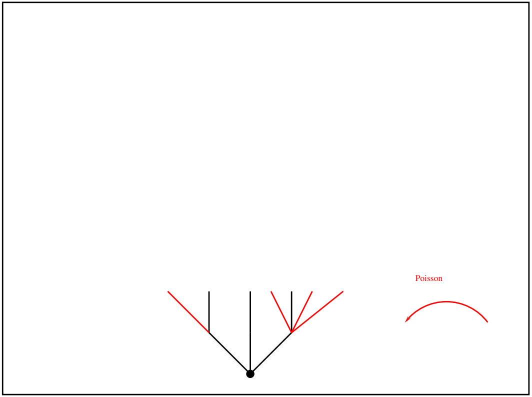
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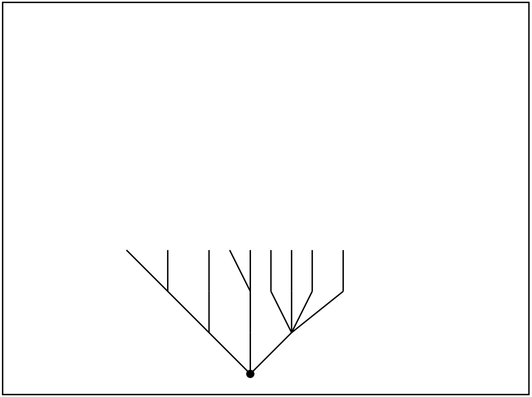

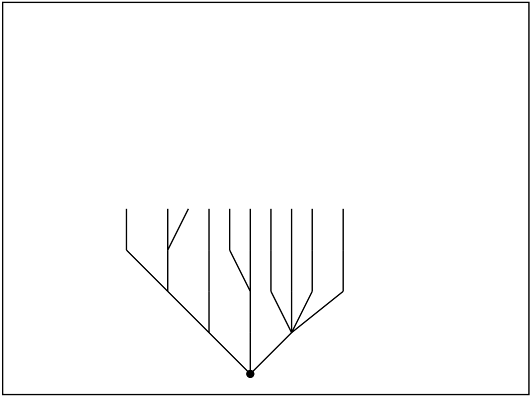

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- Given the the number k of children in the skeleton and the number $n-k$ of grafted trees, the k "immortal" children are chosen uniformly among the n children of u.

Continuity in distribution

Theorem

The family $(\tau^\theta, \theta \in [0, +\infty])$ *is continuous in distribution. In particular*

$$\tau^\theta \xrightarrow[\theta \to 0]{(d)} \tau^0, \qquad \tau^\theta \xrightarrow[\theta \to +\infty]{(d)} \tau^\infty.$$

A combinatorial approach

$\mathbb{T}_N$: set of planar trees with N vertices
$\nu_N^{(\theta)}$ measure on $\mathbb{T}_N$ defined by

$$\nu_N^{(\theta)}(\mathbf{t}) = \frac{1}{W_N^{(\theta)}} e^{\theta H(\mathbf{t})}$$

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Theorem (Durhuus-Ünel, 2023)

Let T_N be distributed according to $\nu_N^{(\theta)}$. Then,

$$T_N \xrightarrow{(d)} \tau^\theta$$

with $\eta = q = 1/2$ i.e. $p(n) = 2^{-(n+1)}$

General offspring distribution

Let p be a **super-critical** offspring distribution.
Let c_n be the Heyde-Seneta normalization: $\frac{1}{c_n} Z_n \to W$.
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If $\lim(a_n/c_n) = \theta$, *then* $(\tau \,|\, Z_n = a_n) \xrightarrow{(d)} \tau^\theta$

- τ^0 *: Kesten*
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Sub-critical case: only if can be obtained as a super-critical GW tree conditioned on extinction.

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Real trees

Definition

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Let e be a continuous function on $[0,\sigma]$ such that $e(0)=e(\sigma)=0$ and $e(t)>0$ for $t\in]0,\sigma[$.

We define a pseudo-metric on $[0,\sigma]$: $d(s,t)=e(s)+e(t)-2\min_{u\in[s,t]}e(u)$ and the equivalence relation $s\sim t \iff d(s,t)=0$.

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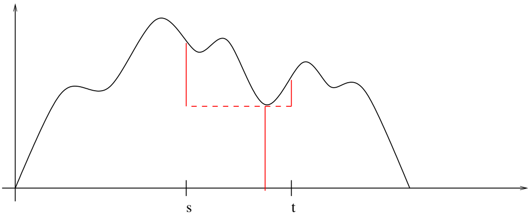
Then the quotient space $([0,\sigma]/\sim, d)$ is a compact real tree rooted at 0 (or σ).

Coding a tree by a continuous function



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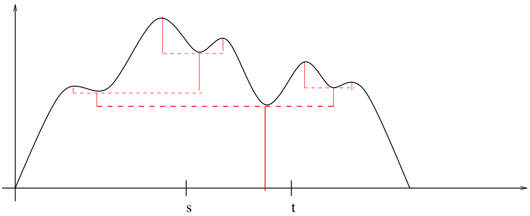

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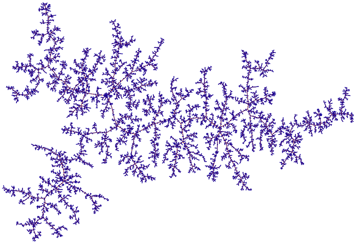
The Brownian tree



thanks to Igor Kortchemski

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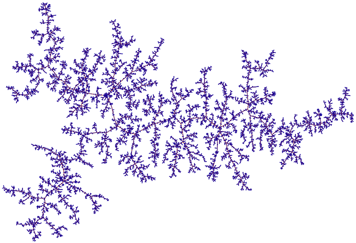

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Theorem (Aldous 1991)

τ_n: critical with finite variance GW tree conditioned on $|\tau_n| = n$.

$$\left(\tau_n, \frac{1}{\sigma\sqrt{n}} d_{gr}\right) \xrightarrow{(d)} \mathcal{T}$$

for the Gromov-Hausdorff topology.

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For $\theta < 0$, we define $\mathbb{N}^\theta$ by

$$\forall h > 0,\ \forall \phi, \qquad \mathbb{N}^\theta[\phi(r_h(\mathcal{T}))] = \mathbb{N}^{|\theta|}[e^{2\theta Z_h}\phi(r_h(\mathcal{T}))]$$

where Z_h is the "local time" of $\mathcal{T}$ at height h.

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$(Z_h, h \geq 0)$: CSBP with branching mechanism

$$\psi_\theta(\lambda) = \frac{1}{2}\lambda^2 + \theta\lambda.$$

Local limit of the conditioned tree

Set $c_h^\theta = \begin{cases} h^2 & \text{if } \theta = 0, \\ \frac{1}{4\theta^2} e^{-|\theta| h} & \text{if } \theta \neq 0. \end{cases}$

Theorem (A.-Delmas-He, 2023+)

Suppose that $a_h > 0$ and $\lim_{h\to+\infty} \frac{a_h}{c_h^\theta} = \alpha \in [0, +\infty)$. Then, under $\mathbb{N}^\theta[\cdot \mid Z_h = a_h]$

$$\mathcal{T} \xrightarrow[h\to+\infty]{(d)} \mathcal{T}^{\alpha,|\theta|}$$

for the local Gromov-Hausdorff topology.

Kesten tree

The tree $\mathcal{T}^{0,|\theta|}$ is distributed as

$$[0,+\infty) \circledast_{i\in I} (h_i, \mathcal{T}_i)$$

where $(h_i, \mathcal{T}_i)_{i\in I}$ are the atoms of Poisson point measure on $\mathbb{R}_+ \times \mathbb{T}$ with intensity

$$dh\, \mathbb{N}^{|\theta|}[d\mathcal{T}].$$

The infinite backbone for $\alpha > 0$

Set $\xi_1 < \xi_2 < \cdots < \xi_n < \cdots$ the atoms of a Poisson point measure on $\mathbb{R}_+$ with intensity $\frac{1}{2}\alpha e^{|\theta| t}$.

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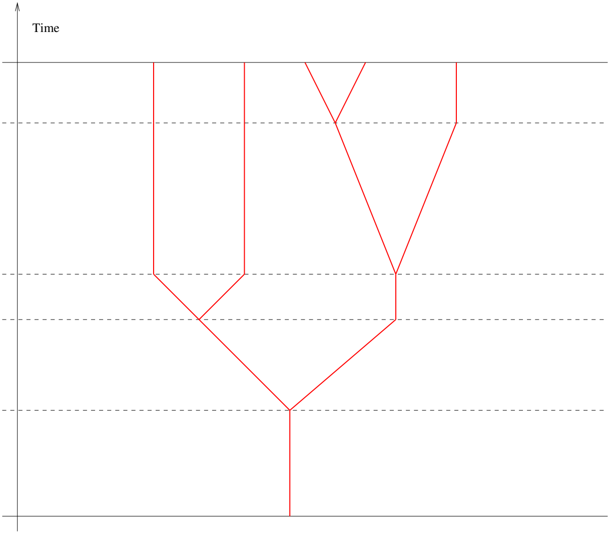
Set $\mathcal{T}^{ske} = \lim_{t \to +\infty} \mathcal{T}_t$.

The infinite backbone for $\alpha > 0$



The tree $\mathcal{T}^{\alpha,|\theta|}$

- $\Lambda(dx)$: length measure on $\mathcal{T}^{ske}$.
- $(x_i, \mathcal{T}_i)_{i\in I}$ atoms of a Poisson-point measure on $\mathcal{T}^{ske} \times \mathbb{T}$ with intensity

$$\Lambda(dx)\mathbb{N}^{|\theta|}[d\mathcal{T}].$$

- $\mathcal{T}^{\alpha,|\theta|} = \mathcal{T}^{ske} \circledast_{i\in I} (x_i, \mathcal{T}_i)$.

Thank you for your attention.