Support properties for SPDE models for spatial branching processes with interaction

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Continuous state branching

Let ψ be the Laplace exponent of a spectrally positive Lévy process:

$$\psi(\lambda) = a\lambda + \frac{\sigma^2}{2}\lambda^2 + \int_{(0,\infty)} \left(\exp^{-\lambda r} - 1 + \lambda r\right) \nu(dr),$$

where $a \in \mathbb{R}$, $\sigma \geq 0$, $\int (r \wedge r^2)\nu(dr) < \infty$.

The ψ-continuous state branching process (CSBP) is the $\mathbb{R}_+$-valued Markov process $(Z_t)_{t \geq 0}$ with Laplace functional

$$\mathbb{E}_a^Z(e^{-\lambda Z_t}) = e^{-a u_\lambda(t)},$$

where $u_\lambda(t)$ solves the evolution equation

$$\dot{u}_\lambda(t) = -\psi(u_\lambda(t)), \quad u_\lambda(0) = \lambda.$$

Super-Brownian motion

A ψ-super-Brownian motion is the $\mathcal{M}_f(\mathbb{R}^d)$ (measure)-valued process $(X_t)_{t\geq 0}$ with Laplace functional

$$\mathbb{E}^X_{X_0}(e^{-\langle X_t,\phi\rangle}) = e^{-\langle X_0, v_\phi(t,\cdot)\rangle},$$

where $v_\phi(t)$ satisfies the PDE

$$\dot{v}_\phi(t,x) = \frac{1}{2}\Delta v_\phi(t,x) - \psi(v_\phi(t,x)), \quad v_\phi(0,x) = \phi(x).$$

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The cases $\psi(u) = u^2$ and $\psi(u) = u^\alpha$, $\alpha \in (0,2)$, correspond respectively to binary and α-stable branching.

(If $\alpha \in (1,2)$, we typically write $\alpha = 1+\beta$, $\beta \in (0,1)$.)

The compact support property

Fact: for $\psi(u) = u^2$ and $\psi(u) = u^\alpha$, $\alpha \in (1, 2)$, the ψ-super-Brownian motion X_t satisfies the **compact support property**:

$$\text{If } X_0 \text{ has compact support,}$$

$$\overline{\bigcup_{s \in [0,t]} \operatorname{supp}(X_s)} \text{ is compact for all } t > 0 \text{ a.s.}$$

CSBP and SDE

Lamperti transform: the ψ-CSBP is the Lévy process with characteristic exponent ψ time-changed to run at speed equal to its size.

Suppose that ψ is the branching mechanism of a pure jump CSBP with Lévy measure ν. Let $N(dr, dz, ds)$ be a compensated Poisson random measure with intensity $\nu(dr)dzds$. We may then write

$$Z_t - Z_0 = \int_0^t \int_0^{Z_{s-}} \int_0^\infty rN(dr, dz, ds).$$

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CSBP and SDE

If $\psi(u) = u^\alpha$, then $\nu(dr) = c_\alpha r^{-1-\alpha}$, and a scaling calculation shows that the following processes are equal in law:

$$\int_0^t \int_0^{Z_{s-}} \int_0^\infty r\, N(dr, dz, ds) \quad \text{and} \quad \int_0^t \int_0^1 \int_0^\infty (Z_{s-}^{1/\alpha} r)\, N(dr, dz, ds).$$

In particular, Z_t satisfies the jump SDE

$$dZ_t = Z_{t-}^{1/\alpha} dW_t,$$

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The CSBP associated to $\psi(u) = u^2$ is Feller's branching diffusion

$$dZ_t = Z_t^{1/2} dB_t.$$

Super-Brownian motion and SPDE

Similarly, for $\psi(u) = u^2$, the density of ψ-super-Brownian motion in $d = 1$ satisfies

$$\partial_t X(t,x) = \frac{1}{2}\Delta X(t,x) + X(t,x)^{1/2}\dot{\xi}_2(t,x), \quad t > 0,\, x \in \mathbb{R}.$$

where $\dot{\xi}_2$ is space-time white Gaussian noise.

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where $\dot{\xi}_2$ is space-time white Gaussian noise.

For $\psi(u) = u^\alpha$, the density of ψ-super-Brownian motion for $d < \frac{2}{\alpha-1}$ satisfies

$$\partial_t X(t,x) = \frac{1}{2}\Delta X(t,x) + X(t,x)^{1/\alpha}\dot{\xi}_\alpha(t,x), \quad t > 0,\, x \in \mathbb{R}^d.$$

where $\dot{\xi}_\alpha$ is space-time white (spectrally positive) α-stable noise.

State-dependent branch rates

In a *non-linear CSBP*, the branch rate is a function of the total population size: for some increasing $R : \mathbb{R}_+ \to \mathbb{R}_+$,

$$Z_t - Z_0 = \int_0^t \int_0^{R(Z_{s-})} \int_0^\infty r N(dr, dz, ds).$$

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In the Brownian and stable cases, we respectively have

$$dZ_t = R(Z_t)^{1/2} dB_t$$

and

$$dZ_t = R(Z_{t-})^{1/\alpha} dW_t.$$

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Decompose noise term as

$$X(t,x)^{\gamma}\dot{\xi}(t,x) = \underbrace{X(t,x)^{\gamma - \frac{1}{\alpha}}}_{\text{variable branch rate}} \times \underbrace{X(t,x)^{\frac{1}{\alpha}}\dot{\xi}(t,x)}_{\alpha\text{-stable branching}}$$

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For what values of γ does $X(t,\cdot)$ have compact support?

Compact support - Gaussian noise.

Let $\alpha = 2$. (Gaussian noise; $d = 1$.) Consider a non-negative solution to

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Theorem (Shiga 1994; Mueller & Perkins, 1992; Krylov, 1997)

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Theorem (Shiga 1994; Mueller & Perkins, 1992; Krylov, 1997)

- *If* $\gamma < 1$, X *has the compact support property.*

Theorem (Mueller, 1990)

- *If* $\gamma \geq 1$, $X(t,x) > 0$ *for all* $t > 0$, $x \in \mathbb{R}$.

Compact support - stable noise (main result)

Let $\alpha \in (1, 2)$. (α-stable noise; $d < \frac{2}{\alpha - 1}$.) Consider a non-negative solution to

$$\partial_t X(t,x) = \frac{1}{2}\Delta X(t,x) + X(t,x)^\gamma \dot{\xi}_\alpha(t,x), \quad t > 0,\, x \in \mathbb{R}^d.$$

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Theorem (H. 2022+)

- *($d = 1$) If $2 - \alpha < \gamma < 1$, X has the compact support property.*
- *($d > 1$) If $1/\alpha \leq \gamma < 1$, X has the compact support property.*

Elements of the proof

The proof uses the method of Krylov.

First let $d = 1$. Suppose $\text{supp}(X_0) \subset (-R, R)$. Mass moves continuously by diffusion, so if $X_t([R, \infty)) > 0$, we must have $A_t(R) > 0$, where

$$A_t(x) = \int_0^t X(s, x) ds.$$

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Integration by parts gives

$$\int_x^\infty (y - x) Y_t(y) dy = A_t(x) + \int_{(0,t] \times \mathbb{R}} (y - x)_+ Y_s(y)^\gamma \xi(ds, dy).$$

Elements of the proof

Use the IBP formula to obtain estimates of the following form: for $x_1 \approx x_0 + r$,

$$\mathbb{P}(A_t(x_1) \geq a) \leq \mathbb{P}(A_t(x_0) \geq b) + Error(a, b, t, r).$$

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Iterating this estimate for a good choice of parameters, one obtains, for some $L > 0$,

$$\mathbb{P}(A_t(x + L) > 0) \leq \mathbb{P}(A_t(x) > \epsilon) + O(\epsilon).$$

Taking $x \to \infty$ and $\epsilon \to 0$, conclude that

$$\lim_{x \to \infty} \mathbb{P}(A_t(x) > 0) = 0.$$

Elements of the proof; other remarks

For $d > 1$: the occupation density $A_t(x)$ is replaced with the occupation density along a $(d-1)$-dimensional hyperplane or sphere. Some complexities arise.

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It is supposed to be easy for small γ - what goes wrong?

A result on explosions

Concerning blow-up, consider the following theorem: let X be a non-negative solution to

$$\partial_t X(t,x) = \frac{1}{2}\Delta X(t,x) + X(t,x)^\gamma \dot{\xi}_2(t,x), \quad t > 0,\ x \in \mathbb{R}.$$

Theorem (Mueller, 2000)

If $\gamma > 3/2$, $\|X_t\|_\infty \to \infty$ in finite time a.s.

Thank you!