Recurrent extensions and stochastic differential equations

Víctor Rivero (CIMAT, Mexico)

Joint work with Sandra Palau and Juan Carlos Pardo

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Feller (1952)

- What are the totality of $\mathbb{R}^+$-valued Markov processes with càdlàg paths such that the process killed at the first hitting time of zero has the same law as a Brownian motion killed at 0 and zero is a regular state?
- Nowadays, What are the totality of recurrent extensions of a Brownian motion?
- What is the stochastic differential equation satisfied by the recurrent extension of a Brownian motion?
- What is the form of its infinitesimal generator?
- More generally, what can be said about a general Feller process in $\mathbb{R}$?

Let B be a standard Brownian motion, Δ a cemetery state

- The minimal process: Brownian motion killed at zero

$$X_t = \begin{cases} B_t, & t < T_0 = \inf\{s > 0 : B_s = 0\}; \\ \Delta, & t \geq T_0. \end{cases}$$

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- Brownian motion absorbed at zero $X_t = B_t 1_{\{t < T_0\}}$.

Reflected Brownian motion

Let B be a standard Brownian motion

- Reflected brownian motion $R_t = B_t - I_t,\ t \geq 0$; with $I_t = \inf_{s \leq t} \{B_s\},\ t \geq 0.$

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- Reflected brownian motion $R_t = B_t - I_t$, $t \geq 0$; with $I_t = \inf_{s \leq t} \{B_s\}$, $t \geq 0$.
  - A version of its local time at zero is the process

$$L_t := -I_t, \qquad t \geq 0,$$

is the unique additive functional that is carried by the set of times where R is at zero.

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  - A version of its local time at zero is the process

    $$L_t := -I_t, \qquad t \geq 0,$$

    is the unique additive functional that is carried by the set of times where R is at zero.
  - Reflected brownian motion and its local time, say (X, L), is the unique weak solution to the stochastic differential equation

    $$dX_t = 1_{\{X_t \neq 0\}} dW_t + dL_t(X), \qquad t \geq 0,$$

    with W a standard brownian motion.

Excursion process

Reflected Brownian motion can be decomposed into its excursions from zero:

- Inverse local time

$$\tau_t = \inf\{s > 0 : L_s > t\}, \quad \tau_{t-} = \lim_{s \uparrow t} \tau_s;$$

- $\overline{\{\tau_t, t > 0\}}$ has the same law as $\overline{\{t > 0 : R_t = 0\}}$;
- (τ_{t-}, τ_t), $t > 0$ are the excursion intervals;
- excursion in local time t

$$\mathbf{e}_t = \begin{cases} R_{\tau_{t-}+s} & 0 < s \leq \tau_t - \tau_{t-}; \\ \Delta, & \tau_t - \tau_{t-} = 0; \end{cases}$$

- $((t, \mathbf{e}_t), t > 0)$ is a Poisson point process on $(0, \infty) \times \mathbb{D}[0, \infty)$ with intensity measure $ds\mathbf{N_0}(d\mathbf{e})$, $\mathbf{N_0}$ is the so-called excursion measure.

Excursion measure

R is a reflected brownian motion

- Under $\mathbf{N_0}$ the coordinate process follows the law of the minimal process:

$$\begin{aligned}\mathbf{N_0}\left(F(\mathbf{e}(s), s \leq t)G(\mathbf{e}(u), u \in [t, \zeta])\right) \\ = \mathbf{N_0}\left(F(\mathbf{e}(s), s \leq t)\mathbb{E}_{\mathbf{e}(t)}\left(G(B_u, u \in [0, T_0])\right)\right).\end{aligned}$$

- $\mathbf{N_0}$ characterizes R.
- $\mathbf{N_0}$ is a Kuznetsov measure with the extra condition

$$\mathbf{N_0}(1 - \exp\{-\zeta\}) < \infty.$$

Sticky brownian motion

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- Sticky brownian motion: take $L(R) = \{L_t(R), t \geq 0\}$ be the local time at zero for the reflected brownian motion R, $\rho > 0$,

$$c_t^{(\rho)} = \inf\{s > 0 : s + \rho L_s > t\}, \qquad t > 0.$$

The process

$$X_t^{(\rho)} = R_{c_t^{(\rho)}}, t \geq 0$$

behaves like R while in $(0, \infty)$ and it is such that a.s.

$$\int_0^{\cdot} 1_{\{X_s^{(\rho)}=0\}} ds = \rho L_t(X^{(\rho)}) > 0,$$

$\forall t > 0.\ a.s.$

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- We will refer to this time change as the "sticky time change".

The effect of the time change is to delay at zero the path of R but the excursions remain the same

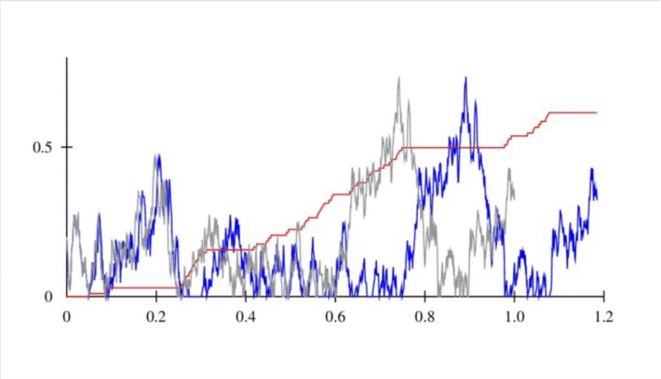

Figure: The path of a sticky brownian motion in blue, built from the path of a reflected brownian motion (in grey), and the local time of the sticky process (in red)

Source: Kostrykin et al. (2010) Researchgate.

Excursion process of sticky brownian motion

The excursion process of the sticky reflected brownian motion is a Poisson point process with the same intensity measure $ds\mathbf{N_0}$ but its inverse local time process

$$\tau_t^{(\rho)} = \inf\left\{s > 0 : L_s(X^{(\rho)}) > t\right\}, \qquad t > 0,$$

is a process with independent and stationary increments with Laplace exponent

$$\begin{aligned} -\frac{1}{t}\log \mathbf{E}(\exp\{-\lambda\tau_t^{(\rho)}\}) = \phi_\rho(\lambda) &= \rho\lambda + \mathbf{N_0}(1-\exp\{-\lambda\zeta\}) \\ &= \rho\lambda + (1-\rho)(\lambda)^{1/2}, \qquad \lambda \geq 0; \end{aligned}$$

and $\phi_\rho(1) = 1$. We always normalize the inverse local time subordinator to satisfy this.

Theorem (See e.j. Lejay's (2006) survey and Salins and Spiliopoulos (2017))

Sticky brownian motion and its local, time at zero, say (X, L), is the unique weak solution to the stochastic differential equation

$$dX_t = 1_{\{X_t \neq 0\}} dW_t + L_t(X)$$

$$\int_0^{\cdot} ds 1_{\{X_s = 0\}} ds = \rho L_{\cdot}(X), \qquad a.s.$$

Feller brownian motion: jumping brings more fun

- Let $\sigma = (\sigma_t, t \geq 0)$ be a subordinator: an strictly increasing process with independent and stationary increments,

$$T_x = \inf\{s > 0 : \sigma_s > x\}, \qquad x \geq 0.$$

Assume σ is independent of B. Define the overshoot process, $O_x = \sigma_{T_x} - x$, $x \geq 0$. The process defined by

$$X_{\sigma,t} := B_t - I_t + O_{-I_t} = B_t + \sigma_{T_{-I_t}}, \qquad t \geq 0,$$

is a process that behaves like a brownian motion while in $(0, \infty)$, and when it hits zero it either returns into $(0, \infty)$ continuously or by a jump which is distributed according to the jump measure of σ.

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- Its sticky time change version makes sense.

Theorem (Feller (1952), Itô and Mckean 1961,...)

Let σ be a subordinator with Laplace exponent

$$\phi(\lambda) = q + a\lambda + \int_{(0,\infty)} (1 - \exp\{-\lambda x\})\Pi(dx), \qquad \lambda \geq 0,$$

with $q \geq 0$, $a > 0$ or $\Pi(0,\infty) = \infty$, and

$$X_{\sigma,t} := B_t - I_t + O_{-I_t} = B_t + \sigma_{T_{-I_t}}, \qquad t \geq 0,$$

with $T_x = \inf\{s > 0 : \sigma_s > x\}$, $x \geq 0$. The ρ-sticky version of the process $X_{\sigma,\cdot}$, say $X_{(\sigma,\rho)}$, is a Feller process for which zero is a regular state, and the infinitesimal generator is given by, for $f : [0,\infty) \mapsto \mathbb{R}, \in C^2(0,\infty)$,

$$Lf(x) = \frac{1}{2}f''(x), x > 0$$

$$\frac{\rho}{2}f''(0+) = -qf(0) + af'(0+) + \int_{(0,\infty)} (f(y) - f(0))\,\Pi(dy).$$

The excursion process of the Feller brownian motion $X_{(\sigma,\rho)}$, is a Poisson point process, whose characteristic or excursion measure is given by

$$a\mathbf{N_0}(\cdot) + \int_{(0,\infty]} \Pi(dy)\mathbb{E}^0_y\left(\cdot\right),$$

with $\mathbb{E}^0_y$ the law of a Brownian motion killed at zero and issued from $y > 0$.
Notice $\Pi(\{\infty\}) = q$.
Its inverse local time process

$$\tau^{(\rho)}_t = \inf\left\{s > 0 : L_s(X_{(\sigma,\rho)}) > t\right\}, \qquad t > 0,$$

has Laplace exponent

$$\begin{aligned}
&-\frac{1}{t}\log \mathbf{E}(\exp\{-\lambda\tau^{(\rho)}_t\}) = \phi_\rho(\lambda)\\
&= \rho\lambda + a\mathbf{N_0}\left(1-\exp\{-\lambda\zeta\}\right) + \int_{(0,\infty]} \Pi(dy)\left(1-\mathbb{E}_y(e^{-\lambda T_0})\right), \qquad \lambda \geq 0;\\
&= \text{sticky} + \text{continuous exit} + \text{exit by a jump}
\end{aligned}$$

and $\phi_\rho(1) = 1$.

By construction

- The reflected process $R = B - I$ admits a $(-I_t, t \geq 0)$ as a local time at zero
- The process

$$X_{\sigma,t} := B_t - I_t + O_{-I_t} = B_t + \sigma_{T_{-I_t}}, \qquad t \geq 0,$$

  is essentially the reflected process with added jumps in the local time scale, the scale of $-I$.
- The jumps of O are those of σ, and these are given by a Poisson measure $\mathcal{M}$ on $(0, \infty) \times (0, \infty)$ with intensity measure $ds\Pi(dy)$.

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- The jumps of O are those of σ, and these are given by a Poisson measure $\mathcal{M}$ on $(0,\infty) \times (0,\infty)$ with intensity measure $ds\Pi(dy)$.
- One can guess that in the SDE description of $X_{(\sigma,\rho)}$ a term

$$\mathcal{M}(dL(X_{(\sigma,\rho)}), dy)$$

  should be added. It has predictable compensator

$$dL(X_{(\sigma,\rho),t})\Pi(dy)$$

Theorem (Palau, Pardo, R. (2023))

Given a measure Π *on* $(0,\infty)$ *with* $\int_{(0,\infty)} y \wedge 1 \Pi(dy) < \infty$, *and* $a, \rho, q \geq 0$ *such that* $q + \rho + a + \int_{(0,\infty)} \Pi(dy) \left(1 - \mathbb{E}_y(e^{-T_0})\right) = 1$, *there exists:*

- *a process* $X_{(\sigma,\rho)}$ *for which* 0 *is a regular state,*
- *an additive functional of* $X_{(\sigma,\rho)}$, $L(X_{(\sigma,\rho)})$, *whose support is the closure of* $\{t > 0 : X_{(\sigma,\rho),t} = 0\}$,
- *a random measure* $\mathcal{M}$ *on* $(0,\infty) \times (0,\infty]$ *with predictable compensator* $dL_s(X_{(\sigma,\rho)})[\Pi(dy) + q\delta_\infty(dy)]$,

such that $X_{(\sigma,\rho)}$ *is the unique weak solution to the stochastic differential equation*

$$dX_{(\sigma,\rho),t} = 1_{\{X_{(\sigma,\rho),t}>0\}} dB_t + a dL_t(X_{(\sigma,\rho)}) + \int_{(0,\infty]} y \mathcal{M}(dL(X_{(\sigma,\rho),t}), dy),$$

and

$$\int_0^{\cdot} 1_{\{X_{(\sigma,\rho),s}=0\}} ds = \rho L_{\cdot}(X_{(\sigma,\rho)}), \qquad a.s.$$

Construction using Itô's synthesis theorem

Let X be a Brownian motion killed at its first hitting time of zero, a measure $\Pi(dy)$ in $(0,\infty)$, such that $\int_{(0,\infty)}\Pi(dy)1\wedge y<\infty$, and parameters $q,\rho,a\geq 0$, normalized so to have

$$q+\rho+a+\int_{(0,\infty)}\Pi(dy)(1-e^{-y})=1.$$

We realize a Poisson point process "of excursions" $((t,\mathbf{e}_t),t>0)$ on $(0,\infty)\times\mathbb{D}(0,\infty)$ with intensity measure $dt\left(a\mathbf{N_0}(\cdot)+\int_{(0,\infty]}\Pi(dy)\mathbb{E}^0_y(\cdot)\right)$ and we build $X_{\sigma,\rho}$ by glueing together the excursions in the local time scale.
This process is a Feller Brownian motion, it is called a recurrent extension of the Brownian motion on $\mathbb{R}^+$ killed at zero, associated to the excursion measure

$$\mathbf{N}=a\mathbf{N_0}(\cdot)+\int_{(0,\infty]}\Pi(dy)\mathbb{E}^0_y(\cdot)$$

Parallel with Continuous state branching processes

CBI's are constructed using a Dynkin measure:

- Realize a Poisson point process of excursions $((t, \mathbf{w}_t), t > 0)$ on $(0, \infty) \times \mathbb{D}(0, \infty)$ with intensity measure $dtQ_H(dw)$, Q_H the Dynkin measure of the CBI. Denote by $\mathcal{Q}(ds, dw)$ the associated Poisson measure
- Under Q_H the coordinate process behaves as a CBI killed at zero.
- $$Y_t = \int_0^t \int_{\mathbb{D}(0,\infty)} w(t-s)\mathcal{Q}(ds, dw), \ t \geq 0,$$

Recurrent excursions are constructed in a similar way. Take an excursion measure **N**, under it the coordinate process behaves as the minimal process.

- Realize a Poisson point process of excursions $((t, \mathbf{e}_t), t > 0)$ on $(0, \infty) \times \mathbb{D}(0, \infty)$ with intensity measure $dtN(de)$, N the excursion measure. Denote by $\mathcal{N}(ds, dw)$ the associated Poisson measure.
- Let ζ_s be the length of the excursion $\mathbf{e}_s$, and

$$\tau_t = \rho t + \sum_0^t \zeta_s, \quad t \geq 0;$$

finiteness is guaranteed by $\mathbf{N}(1 - \exp\{-\zeta\}) < \infty$.

- Define the local time by

$$L_t = \inf\{u > 0 : \tau_u > t\}, \qquad t > 0.$$

- The recurrent extension is

$$X_t^* = \int_0^t \int_{\mathbb{D}(0,\infty)} \mathbf{e}(t - s)\mathcal{N}(dL_s, d\mathbf{e}), \; t \geq 0.$$

Lemma (A martingale problem)

We have the following equivalence

- $X_{(\sigma,\rho)}$ *is a recurrent extension of BM killed at zero associated to the excursion measure* $a\mathbf{N_0}(\cdot) + \int_{(0,\infty]} \Pi(dy)\mathbb{E}^0_y(\cdot)$ *and to the stickyness parameter* ρ*, with* $q + \rho + a + \int_{(0,\infty)} \Pi(dy)(1 - e^{-y}) = 1.$

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We have the following equivalence

- $X_{(\sigma,\rho)}$ *is a recurrent extension of* *BM killed at zero* *associated to the excursion measure* $a\mathbf{N}_0(\cdot)+\int_{(0,\infty]}\Pi(dy)\mathbb{E}^0_y(\cdot)$ *and to the stickyness parameter* ρ*, with* $q+\rho+a+\int_{(0,\infty)}\Pi(dy)(1-e^{-y})=1.$
- *There exists a couple* $(X_{(\sigma,\rho)},L(X_{(\sigma,\rho)}))$*,* 0 *is regular,* $L(X_{(\sigma,\rho)})$ *is an additive functional carried by the set of zeros of* $X_{(\sigma,\rho)}$*, such that*

$$M^f_t=f(X_{(\sigma,\rho),t})-\int_0^t\frac{1}{2}f''(X_{(\sigma,\rho),s})1_{\{X_{(\sigma,\rho),s}\neq 0\}}ds$$

$$-L_t(X_{(\sigma,\rho)})\left(af'(0+)+\int_{(0,\infty]}\Pi(dy)(f(y)-f(0))\right),\ t\geq 0,$$

is a martingale; for any $f:[0,\infty)\mapsto\mathbb{R},\in C^2(0,\infty)$*, with* $f(0)=0$*,* $\frac{\rho}{2}f''(0+)=-qf(0)+af'(0+)+\int_{(0,\infty)}(f(y)-f(0))\Pi(dy)$*; and also*

$$\rho L_\cdot(X_{(\sigma,\rho)})=\int_0^\cdot 1_{\{X_{(\sigma,\rho),s}=0\}}ds,\qquad a.s.$$

Consider Z, a (ϕ, ψ)-continuous state branching process with immigration associated to the branching

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} (e^{-\lambda r} - 1 + \lambda r 1_{\{|r|<1\}})\Lambda(dr),$$

and immigration

$$\phi(\lambda) = a\lambda + \int_{(0,\infty)} \Pi(dy)(1 - \exp\{-\lambda y\}).$$

mechanisms, i.e.

$$\mathbb{E}(\exp\{-\lambda Z_t\}|Z_0 = x) = \exp\{-x u_t(\lambda) - \int_0^t \phi(u_s(\lambda))ds\}$$

$u_t(\lambda) + \int_0^t \psi(u_s(\lambda))ds = \lambda$, $\lambda, t \geq 0$. Lambert(2002) built the height process of Z as the local time at zero of the *genealogy coding process* X^*. We know a SDE for Z, what is the SDE satisfied by X^*?

- Let X be a spectrally positive Lévy process with Laplace exponent

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} (e^{-\lambda r} - 1 + \lambda r 1_{\{|r|<1\}})\Lambda(dr),$$

  and $I_t = \inf\{X_s, s \leq t\}$, $t \geq 0$.

- Let σ be a subordinator independent of X, and with Laplace exponent

$$\phi(\lambda) = a\lambda + \int_{(0,\infty)} (1 - \exp\{-\lambda x\})\Pi(dx), \qquad \lambda \geq 0,$$

  with $a > 0$ or $\Pi(0,\infty) = \infty$.

- Define $T_x = \inf\{s > 0 : \sigma_s > x\}$, $x \geq 0$, and the overshoot process, $O_x = \sigma_{T_x} - x$, $x \geq 0$.

The genealogy coding process, defined by

$$X^*_t := X_t - I_t + O_{-I_t} = X_t + \sigma_{T_{-I_t}}, \qquad t \geq 0,$$

is a strong Markov process, whose local time at zero is given by $\sigma_{T_{-I_\cdot}}$.

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is a strong Markov process, whose local time at zero is given by $L^*_\cdot = \sigma_{T_{-I_\cdot}}$. The height process is defined as

$$H_t = L^*_t + \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_{g_t}^t 1_{\{X^*_s - \inf_{s \leq u \leq t}\{X^*_u\} < \epsilon\}} ds, \qquad t > 0,$$

$g_t = \sup\{u < t : X^*_u = 0\}$. Lambert proved that the random measure

$$h \mapsto \int_0^\infty h(H_t)dt,$$

has a.s. a càdlàg density $(\mathcal{Z}_a, a \geq 0)$ w.r.t. Lebesgue measure and this is a (ϕ, ψ)-CBI.

Theorem (Palau, Pardo, R. (2023))

Let X be a spectrally negative Lévy process. Given a measure Π on $(0,\infty)$ with $\int_{(0,\infty)} y \wedge 1 \Pi(dy) < \infty$, and a such that $a + \int_{(0,\infty)} \Pi(dy) \left(1 - \mathbb{E}_y(e^{-T_0})\right) = 1$, there exists:

- *a process X^* for which 0 is a regular state,*
- *an additive functional of X^*, L^*, whose support is the closure of $\{t > 0 : X^*_t = 0\}$,*
- *a random measure $\mathcal{M}$ on $(0,\infty) \times (0,\infty]$ with predictable compensator $dL^*_s \Pi(dy)$,*

such that the genealogy coding process associated to X and immigration mechanism ϕ, say X^*, is the unique weak solution to the stochastic differential equation

$$dX^*_t = 1_{\{X^*_t > 0\}} dX_t + a dL^*_t + \int_{(0,\infty]} y \mathcal{M}(dL^*_t, dy), t \geq 0.$$

Take X a Feller process with infinitesimal generator $\mathcal{L}$, and assume that it admits an extension X^* with respect to an excursion measure $\mathbf{N}$. The remark that the recurrent extension is

$$X^*_t = \int_0^t \int_{\mathbb{D}(0,\infty)} \mathbf{e}(t-s)\mathcal{N}(dL_s, d\mathbf{e}), \ \ t \geq 0,$$

allows to prove that

- for $f \in \mathcal{D}(\mathcal{L}^*)$ the process

$$f(X^*_t) - \int_0^t \mathcal{L}f(X^*_s)1_{\{X^*_s \neq 0\}} ds + L_t \left(\mathbf{N}\left(\int_0^\zeta \mathcal{L}f(\mathbf{e}_u)du\right)\right), \qquad t \geq 0,$$

  is a martingale.

From here one can deduce SDE.
And also the fact that the measure

$$m(dy) = \rho\delta_0(dy) + \mathbf{N}\left(\int_0^\zeta 1_{\{\mathbf{e}_s \in dy\}} ds\right)$$

is invariant when $q = 0$, and that

$$m\mathcal{L}f = 0,$$

derive the boundary condition for the infinitesimal generator.

Feller difussions

Theorem (Palau, Pardo, R. (2023))

Assume the minimal process X satisfies the equation

$$dX_t = \sigma(X_t)1_{\{X_t \neq 0\}} dB_t, \qquad X_0 = x,$$

admits a recurrent extension that leaves zero continuously. Given a measure Π on $(0,\infty)$ with $\int_{(0,\infty)}(y \wedge 1)\Pi(dy) < \infty$, there exists:

- *a process $X_{(\sigma,\rho)}$ for which 0 is a regular state,*
- *an additive functional of $X_{(\sigma,\rho)}$, $L(X_{(\sigma,\rho)})$, whose support is the closure of $\{t > 0 : X_{(\sigma,\rho),t} = 0\}$,*
- *a random measure $\mathcal{M}$ on $(0,\infty) \times (0,\infty]$ with predictable compensator $dL_s(X_{(\sigma,\rho)})[\Pi(dy) + q\delta_\infty(dy)]$,*

such that $X_{(\sigma,\rho)}$ is the unique weak solution to the stochastic differential equation

$$dX_{(\sigma,\rho),t} = \sigma(X_{(\sigma,\rho),t})1_{\{X_{(\sigma,\rho),t} \neq 0\}} dB_t + a dL_t(X_{(\sigma,\rho)}) + \int_{(0,\infty]} y \mathcal{M}(dL(X_{(\sigma,\rho),t}), dy),$$

and $\int_0^{\cdot} 1_{\{X_{(\sigma,\rho),s}=0\}} ds = \rho L_{\cdot}(X_{(\sigma,\rho)}),$ *a.s.*

Thank you for your attention!

Merci pour votre attention!

¡Gracias por su atención!
¡Vamos a comer!