The explosion of continuous-state nonlinear branching processes with big jumps

Xiaowen Zhou

Concordia University

Branching Processes and Applications

May 22-26, 2023, Angers

Based on joint work with Clement Foucart and Bo Li

Outline of the Talk

Continuous-state branching process (CSBP)

- A continuous-state branching process is a nonnegative Markov process X (with no negative jumps) satisfying the additive branching property,

$$\mathbb{E}_{x+y}e^{-\theta X_t} = \mathbb{E}_x e^{-\theta X_t}\mathbb{E}_y e^{-\theta X_t}, \ \ x, y > 0.$$

- Its Laplace transform is determined by

$$\mathbb{E}_x e^{-\theta X_t} = e^{-xu_t(\theta)}$$

where function $u_t(\theta)$ satisfies $\frac{\partial u_t(\theta)}{\partial t} + \psi(u_t(\theta)) = 0$ with $u_0(\theta) = \theta$ and branching mechanism

$$\psi(\lambda) = b\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_0^\infty (e^{-\lambda x} - 1 + \lambda x)\Pi(\mathrm{d}x)$$

for $\sigma > 0, b \in \mathbb{R}$ and ν satisfying $\int_0^\infty (1 \wedge z^2)\nu(\mathrm{d}z) < \infty$.
- X is critical, subcritical, supercritical if $\mathbb{E}X_t$ is constant, decreasing or increasing, respectively.
- It is the total mass process of super-Brownian motion.

CSBP as solution to SDE

In general, CSBP is the unique nonnegative solution to

$$
\begin{aligned}
X_t = & X_0 + \int_0^t bX_s \mathrm{d}s + \int_0^t \int_0^{\gamma X_s} W(\mathrm{d}s, \mathrm{d}u) \\
& + \int_0^t \int_0^{X_s} \int_0^\infty z\tilde{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}z),
\end{aligned}
$$

where $W(\mathrm{d}s, \mathrm{d}u)$ is a time-space white noise and and $\tilde{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}z)$ is an independent compensated spectrally positive Poisson random measure on $(0, \infty)^3$ with intensity $\mathrm{d}s\mathrm{d}u\nu(\mathrm{d}z)$.
Solution to the above equation has the same distribution as that to

$$
\begin{aligned}
X_t = & X_0 + \int_0^t bX_s \mathrm{d}s + \int_0^t \sqrt{\gamma X_s} \mathrm{d}B_s \\
& + \int_0^t \int_0^{X_s} \int_0^\infty z\tilde{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}z).
\end{aligned}
$$

where B_s is a Brownian motion.

Why CSBP?

- The relation between BGW branching process and CSBP is like that between Markov chain and Brownian motion.
- On one hand, CSBPs keep the key common features and ignore the minor differences between the discrete-state branching processes.
- On the other hand, by considering the continuous-state processes we can apply Lévy process and SDE theories.

Why is CSBP spectrally positive?

- The continuous-time discrete-state branching process has negative jumps of size 1, which disappears in the scaling limit.

Spectrally positive Lévy process (SPLP)

- A spectrally positive Lévy process ξ has stationary independent increments and no negative jumps.
- For $\xi_0 = 0$,
$$\mathbb{E}\mathrm{e}^{-\lambda\xi_t} = \mathrm{e}^{t\psi(\lambda)},$$
for $\lambda, t \geq 0$, where the Laplace exponent for $-\xi$ is
$$\psi(\lambda) = b\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{-\infty}^{0} \left(\mathrm{e}^{\lambda z} - 1 - \lambda z\right) \nu(\mathrm{d}z),$$
for $b \in \mathbb{R}$ and $\sigma \geq 0$. We assume that the σ-finite Lévy measure ν on $(-\infty, 0)$ satisfies $\int_{-\infty}^{0}(1 \wedge z^2)\nu(\mathrm{d}z) < \infty$.
- The scale function W of the process $-\xi$ is defined as the function with Laplace transform on $[0, \infty)$ given by
$$\int_0^\infty \mathrm{e}^{-\lambda z} W(z)\mathrm{d}z = \frac{1}{\psi(\lambda)} \quad \text{for } \lambda > \Phi(0) := \psi^{-1}(0).$$

Lamperti transform

- CSBP is associated with SPLP via the Lamperti time change.
- Write ξ for a spectrally positive Lévy process and $\tau_0^- := \inf\{t : \xi_t = 0\}$ for its first time of reaching 0. Write

$$\eta(t) := \int_0^{t\wedge\tau_0^-} \frac{1}{\xi_s} ds \quad \text{and} \quad \eta^{-1}(t) := \inf\{s \geq 0 : \eta(s) > t\}.$$

  Then process $X_t := \xi_{\eta^{-1}(t)\wedge\tau_0^-}$ is the CSBP.
- Loosely speaking, process X_t runs through the sample path of ξ_t at a different speed. X_t speeds up when $R(X_t)$ takes large values and slows down when $R(X_t)$ takes small values.

A class of continuous-state nonlinear branching processes

- We can generalize the Lamperti transform to introduce nonlinear CSBPs. Let R be a positive continuous function on $(0, \infty)$ satisfying $\inf_{x>\epsilon} R(x) > 0, \forall \epsilon > 0$.
- Define

$$\eta(t) := \int_0^{t\wedge\tau_0^-} \frac{1}{R(\xi_s)} ds, \quad \eta^{-1}(t) := \inf\{s < \tau_0^- : \eta(s) > t\},$$

Let

$$X_t := \begin{cases} \xi_{\eta^{-1}(t)\wedge\tau_0^-}, & t < \eta(\tau_0^-), \\ \xi_{\eta(\tau_0^-)-}, & t \geq \eta(\tau_0^-), \end{cases}$$

be a continuous-state nonlinear branching process with branching rate function $R(\cdot)$.
- Such nonlinear CSBPs was first introduced by P.-S. Li (2019).

Different ways of characterizing nonlinear CSBP

- Process X has a generator L on $C^2(0,\infty)$ such that

$$
\begin{aligned}
&Lf(x) := R(x)L^*f(x)\\
&= R(x)\left(bf'(x) + \frac{\gamma}{2}f''(x) + \int_0^\infty (f(x+u) - f(x) - uf'(x))\nu(du)\right)
\end{aligned}
$$

  where L^* is generator for a SPLP and $f(0) = 0$.
- It arises as the unique nonnegative solution (stopped at 0) to

$$
\begin{aligned}
X_t =& X_0 + \int_0^t bR(X_s)\mathrm{d}s + \int_0^t \sqrt{\gamma R(X_s)}\mathrm{d}B_s\\
&+ \int_0^t \int_0^{R(X_s)} \int_0^\infty z\tilde{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}z).
\end{aligned}
$$

Why nonlinear branching mechanism?

- The nonlinear CSBP allows the branching rate to depend on the current population size and can be a more flexible population model.
- The nonlinear branching mechanism allows the process to have richer asymptotic behaviors.
    - The process comes down from infinity if "starting from ∞", it becomes finite at any positive time, which can not happen to (linear) CSBPs.
    - Explosion occurs if the process approaches to $+\infty$ in finite time, which can not happen to CSBP if its large jumps have finite mean.
    - The process becomes extinguishing if it converges to 0 but never reaches 0, which happens to CSBP in very special cases.

On the other hand, the additive branching property and the associated classical techniques fail in handing nonlinear CSBP. We need to introduce new techniques in its study.

How to reach infinity in finite time (explosion)?

- Let $T_x^+ := \inf\{t : X_t \geq x\}$ and $T^\infty := \lim_{x\to\infty} T_x^+$ be the explosion time. Observe that

$$T^\infty = \begin{cases} \int_0^\infty \frac{1}{R(\xi_s)} \mathrm{d}s & \text{if } \tau_0^- = \infty, \\ \infty & \text{if } \tau_0^- < \infty. \end{cases}$$

- For explosion to happen to X, we need
  - $\xi_t \to \infty$ as $t \to \infty$ (X is supercritical);
  - $R(x) \to \infty$ fast enough as $x \to \infty$ so that $T_\infty^+ < \infty$.
- Suppose that $R(x)$ goes to ∞ fast enough as $x \to \infty$. If X has reached a high level x, then due to the time change, it will speed up on its way to ∞ and can even reach ∞ within finite time, i.e. explosion occurs.
- We are not aware of any previous results on speed of explosion for general Markov processes. Some related results for CSBPs in random environment can be found in Boinghoff and Hutzenthaler (2012) and Palau and Pardo (2016).

The goal and the approach

- There are different ways of characterizing speed of explosion in terms of "how fast X_s grows as $s \to T_\infty^+$".
  - Given $T_\infty^+ < \infty$, $X(T_\infty^+ - t) \to \infty$ as $t \to 0+$.
  - We want to know how fast $X(T_\infty^+ - t)$ increases as $t \to 0+$.
  - A closely related problem is how fast the upcrossing time T_x^+ approaches to T_∞^+ as $x \to \infty$, or how fast the residual explosion time $T_\infty^+ - T_x^+ \to 0$ as $x \to \infty$.
- We use some ideas from Foucart, Li and Z. (2021) for coming down from infinity.
- A main difficulty comes from overshoot when X first upcrosses a level. Such a difficulty does not appear in studying CDI.

Nonlinear CSBP with small jumps

- (Explosion criterion) For $\gamma := \mathbb{E}(\xi_1 - \xi_0) \in (0, \infty)$, explosion occurs for X if and only if $\int^\infty \frac{1}{R(y)} \mathrm{d}y < \infty$; see Döring and Kyprianou (2016).
- CSBP with $R(x) = x$ and $\gamma < \infty$ can not explode.
- Under the above conditions, the overshoot of X when upcrossing a level has an asymptotic stationary distribution,

$$\mathbb{P}(X(T_y^+) - y \in \mathrm{d}z) = \mathbb{P}(\xi(\tau_y^+) - y \in \mathrm{d}z) \Rightarrow \rho(\mathrm{d}z) \quad \text{as } y \to \infty,$$

where $\int_{0-}^\infty e^{-sz} \rho(\mathrm{d}z) = \frac{\Phi(0)\psi(s)}{\gamma s(s - \Phi(0))}$.

Define $\mathbb{P}_1^{\uparrow}(\cdot) := \mathbb{P}_1\big(\cdot \,\big|\, T_\infty^+ < \infty\big)$. Recall $\varphi(x) := \frac{1}{\gamma}\int_x^\infty \frac{dy}{R(y)}$.
Function g is regularly varying with index γ if $g(\lambda x)/g(x) \sim \lambda^\gamma$.
Write $\mathcal{R}_\gamma$ for the set of regular varying functions with index γ.

Theorem

(Li and Z. EJP 2021) If $R \in \mathcal{R}_\beta$ for $\beta > 1$, we have

- $$\frac{T_\infty^+ - T_x^+}{\varphi(x)}\bigg|_{\mathbb{P}_1(\cdot|T_\infty^+<\infty)} \to 1 \quad \text{in probability as } x \to \infty;$$
- $$\frac{X(T_\infty^+ - t)}{\varphi^{-1}(t)}\bigg|_{\mathbb{P}_1(\cdot|T_\infty^+<\infty)} \to 1 \quad \text{in probability as } t \to 0+.$$

In this case the process explodes along a deterministic curve.

Nonlinear CSBPs with large jumps

- In the above results on the speed of explosion, a key assumption is that the big jumps have a finite first moment, which guarantees stationary overshoot.
- The above results can not be applied to (linear) CSBP. Recall the Grey condition for explosion: explosion happens to CSBP if and only if $\int_{0+} 1/|\psi(x)|dx = \infty$. For explosion to happen, the associated SPLP is necessary to have big jumps of infinite mean, i.e. $\int_{-\infty}^{-1} |x|\nu(dx) = \infty$.
- We recently consider the speed of explosion for nonlinear CSBP with big jumps.
- (Explosion criterion) If $\xi_t \to \infty$ and $R(x) \uparrow \infty$, then $\mathbb{P}_x(T_\infty^+ < \infty) > 0$ if and only if for all $x > a > 0$,

$$\mathbb{E}_x[\eta(\infty); \tau_a^- = \infty] = \int_a^\infty \frac{1}{R(y)} \left(e^{\Phi(0)(a-x)} W(y-a) - W(y-x)\right) dy < \infty.$$

Assumptions for nonlinear CSBPs with large jumps

- To handle the large overshoot, we want to show the rescaled overshoot converges for which we need to impose some scaling properties on rate function R and Laplace exponent ψ.
- In the rest of the talk, we assume that $\psi'(0) = -\infty$, $R \in \mathcal{R}_\beta$ at ∞ and $-\psi \in \mathcal{R}_\alpha$ at 0 for some $\beta \geq \alpha$ and $\alpha \in [0,1]$.
- We first consider first the case $\beta > \alpha \geq 0$ in which explosion happens.
- Define for large x

$$\varphi(x) := \int_x^\infty \frac{-1}{yR(y)\psi(1/y)}\,dy \in \mathcal{R}_{\alpha-\beta}, \tag{1}$$

$$\varphi^{-1}(t) := \inf\{r > 0, \varphi(r) = t\}, \quad t > 0.$$

Then

$$\varphi(x) \sim \frac{1}{\beta - \alpha}\frac{-1}{R(x)\psi(1/x)} \in \mathcal{R}_{\alpha-\beta}.$$

Theorem

(Foucart, Li and Z. 2023+) For $\beta > \alpha \geq 0$ we have

$$\frac{T_\infty^+ - T_x^+}{\varphi(x)}\Big|_{\mathbb{P}_1(\cdot|T_\infty^+<\infty)} \Rightarrow \chi_\alpha^{\alpha-\beta} \times \varrho$$

where ϱ has Laplace transform

$$\mathbb{E}[e^{\theta\varrho}] = 1 + \sum_{n\geq 1} \frac{\theta^n}{n!}\Big(\prod_{k=1}^{n} \frac{\Gamma(k(\beta-\alpha)+1)}{\Gamma(k(\beta-\alpha)+\alpha)}\Big) \quad \text{for all } \theta \in \mathbb{R}$$

and where $\chi_\alpha \geq 1$ is a random variable independent to ϱ with

$$\mathbb{P}(\chi_\alpha > t) = \frac{\sin \alpha\pi}{\pi} \int_0^1 u^{\alpha-1}(t-u)^{-\alpha}\, du \quad \text{for } t \geq 1.$$

In particular, $\chi_\alpha = \varrho = 1, \chi_\alpha^{\alpha-\beta} \times \varrho = 1$ if $\alpha = 1$ and $\chi_\alpha = \infty, \varrho$ is exponential, $\chi_\alpha^{\alpha-\beta} \times \varrho = 0$ if $\alpha = 0$.

Recall that the above result holds for CSBP ($\beta = 1$)

Outline of the approach

Given $T_\infty^+ < \infty$, we express $T_\infty^+ - T_x^+$ in terms of ξ, then apply the Markov property at τ_x^+.

$$\begin{aligned}\frac{T_\infty^+ - T_x^+}{\varphi(x)} &= \frac{1}{\varphi(x)} \int_{\tau_x^+}^\infty \frac{1}{R(\xi(t))}\, dt \\ &= \frac{\eta(\infty) \circ \theta_{\tau_x^+}}{\varphi(x)} \\ &= \frac{\varphi(\xi(\tau_x^+))}{\varphi(x)} \times \frac{\eta(\infty) \circ \theta_{\tau_x^+}}{\varphi(\xi(\tau_x^+))},\end{aligned}$$

where $\frac{\varphi(\xi(\tau_x^+))}{\varphi(x)}$ concerns the asymptotic overshoot and $\frac{\eta(\infty)\circ\theta_{\tau_x^+}}{\varphi(\xi(\tau_x^+))}$ concerns the residual explosion time after the first passage time τ_x^+.

We first show that the overshoot after re-scaling converges, which generalizes a result of Bertoin for subordinator.

Proposition

Suppose that $-\psi \in \mathcal{R}_\alpha$ at 0 for some $\alpha \in (0, 1)$. Then

$$x^{-1}\xi(\tau_x^+) \Rightarrow \chi_\alpha \quad \text{as} \quad x \to \infty.$$

If $\alpha = 1$, then $x^{-1}\xi(\tau_x^+) \Rightarrow 1$, and if $\alpha = 0$, then $x^{-1}\xi(\tau_x^+) \Rightarrow \infty$.

Outline of proof: Observe that the overshoot of ξ coincides with the overshoot of its ladder height process H, which is a subordinator with Laplace exponent

$$\Phi(s) = \frac{\psi(s)}{s-p} = \frac{\sigma^2}{2}s + \int_0^\infty \int_0^z \left(e^{-pz+py} - e^{-pz+(p-s)y}\right)\nu(dz)dy$$

$$= \frac{\sigma^2}{2}s + \int_0^\infty (1-e^{-sz})\left(\int_z^\infty e^{p(z-y)}\nu(dy)\right)dz$$

$$=: \frac{\sigma^2}{2}s + \int_0^\infty (1-e^{-sz})\Pi(dz) = s\left(\frac{\sigma^2}{2} + \int_0^\infty e^{-sz}\bar{\Pi}(z)dz\right),$$

where $\bar{\Pi}(z) := \Pi[z,\infty) = \int_z^\infty \Pi(du) \in \mathcal{R}_{-\alpha}$ at ∞.
Denote by $\mathcal{U}$ the renewal measure of H,

$$\mathcal{U}(x) := \mathbb{E}\left[\int_0^\infty \mathbf{1}\big(H(t)\le x\big)dt\right].$$

It is shown by Bertoin that for $0 < u < 1$,

$$\mathbb{P}\Big(\frac{1}{x}\sup_{t<\tau_x^+}\xi_t \in \mathrm{d}u\Big) = \mathcal{U}(x\mathrm{d}u)\bar{\Pi}(x(1-u)) \to \frac{\sin(\alpha\pi)}{\pi}u^{\alpha-1}(1-u)^{-\alpha}\mathrm{d}u.$$

Then

$$\mathbb{P}\Big(\frac{1}{x}\sup_{t<\tau_x^+}\xi_t \in \mathrm{d}u,\ \frac{1}{x}\xi(\tau_x^+ > z)\Big) = \mathcal{U}(x\mathrm{d}u)\bar{\Pi}(x(z-u))$$

$$= \mathcal{U}(x\mathrm{d}u)\bar{\Pi}(x(1-u))\frac{\bar{\Pi}(x(z-u))}{\bar{\Pi}(x(1-u))}$$

$$\to \frac{\sin(\alpha\pi)}{\pi}u^{\alpha-1}(z-u)^{-\alpha}\mathrm{d}u.$$

For the limit of $\frac{\eta(\infty)\circ\theta_{\tau_x^+}}{\varphi(\xi(\tau_x^+))}$ we prove the following result.

Proposition

Suppose $R \in \mathcal{R}_\beta$ at ∞ and $-\psi \in \mathcal{R}_\alpha$ at 0 with $\beta > \alpha \geq 0$, we have

$$\frac{T_\infty^+}{\varphi(x)}\bigg|_{\mathbb{P}_x(\cdot|T_\infty^+<\infty)} \Rightarrow \varrho \quad \text{as } x \to \infty.$$

Outline of proof: For any $f \in \mathcal{R}_{-\gamma}$ at ∞ with $\gamma > \alpha \geq 0$, let

$$U_0 f(x) := \int_0^\infty \mathbb{E}_x[f(\xi_t); t < \tau_0^-]\mathrm{d}t \sim \frac{\Gamma(\gamma-\alpha)}{\Gamma(\gamma)} \frac{-f(x)}{\psi(1/x)}.$$

Let

$$m_n(x) := \mathbb{E}_x[\eta^n(\infty); \tau_0^- = \infty].$$

Then

$$m_1(x) \sim \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)} \frac{-1}{R(x)\psi(1/x)} \sim \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)}\varphi(x), \;\; x \to \infty.$$

$$m_n(x) = n\int_0^\infty \frac{m_{n-1}(y)}{R(y)} U_0(x, \mathrm{d}y) \sim \frac{\Gamma(n(\beta-\alpha)+1)}{\Gamma(n(\beta-\alpha)+\alpha)}\varphi(x)m_{n-1}(x).$$

It follows that

$$\mathbb{E}_x\Big[\Big(\frac{\eta(\infty)}{\varphi(x)}\Big)^n; \tau_0^- = \infty\Big] \to \prod_{k=1}^n \frac{\Gamma(k(\beta-\alpha)+1)}{\Gamma(k(\beta-\alpha)+\alpha)}.$$

Proof of the main theorem

By the previous Propositions,

$$\frac{\varphi(\xi(\tau_x^+))}{\varphi(x)} \sim \varphi\left(\frac{\xi(\tau_x^+)}{x}\right) \Rightarrow \chi_\alpha^{\alpha-\beta} \quad \text{as } x \to \infty$$

and

$$\frac{\eta(\infty) \circ \theta_{\tau_x^+}}{\varphi(\xi(\tau_x^+))}\Big|_{\mathbb{P}_x(\cdot|T_\infty^+<\infty)} \Rightarrow \varrho \quad \text{as } x \to \infty.$$

Then

$$\frac{T_\infty^+ - T_x^+}{\varphi(x)}\Big|_{\mathbb{P}_1(\cdot|T_\infty^+<\infty)} \Rightarrow \chi_\alpha^{\alpha-\beta} \times \varrho.$$

Theorem

Suppose $\beta > \alpha > 0$*, we have*

$$\frac{X(T_\infty^+ - t)}{\varphi^{-1}(t)}\Big|_{\mathbb{P}_1(\cdot|T_\infty^+<\infty)} \Rightarrow \frac{1}{\chi_\alpha} \times \varrho^{\frac{1}{\beta-\alpha}}.$$

In particular, if $\alpha = 1$*, then Suppose* $\beta > \alpha > 0$*, we have*

$$\frac{X(T_\infty^+ - t)}{\varphi^{-1}(t)}\Big|_{\mathbb{P}_1(\cdot|T_\infty^+<\infty)} \to 1.$$

Outline of proof

For $\alpha \in (0,1)$, For $\lambda > 0$ and small $t > 0$,

$$\{\bar{X}(T_\infty^+ - t) > \lambda\varphi^{-1}(t)\} \subset \{T_\infty^+ - t \geq T^+_{\lambda\varphi^{-1}(t)}\} \subset \{\bar{X}(T_\infty^+ - t) \geq \lambda\varphi^{-1}(t)\}$$

where $\bar{X}(T_\infty^+ - t) \sim X(T_\infty^+ - t)$. Then

$$\mathbb{P}_1\Big(T_\infty^+ - T^+_{\lambda\varphi^{-1}(t)} \geq t \Big| T_\infty^+ < \infty\Big) = \mathbb{P}_1\Big(\frac{T_\infty^+ - T^+_{\lambda\varphi^{-1}(t)}}{\varphi(\lambda\varphi^{-1}(t))} \geq \frac{\varphi(\varphi^{-1}(t))}{\varphi(\lambda\varphi^{-1}(t))} \sim \lambda^{\beta-\alpha} \Big| T_\infty^+ < \infty\Big).$$

$$\mathbb{P}_1\Big(\frac{X(T_\infty^+ - t)}{\psi^{-1}(t)} \geq \lambda \Big| T_\infty^+ < \infty\Big) \sim \mathbb{P}_1\Big(T_\infty^+ - T^+_{\lambda\varphi^{-1}(t)} \geq t \Big| T_\infty^+ < \infty\Big) \to \mathbb{P}\big(\chi_\alpha^{\alpha-\beta}\varrho \geq \lambda^{\beta-\alpha}\big) = \mathbb{P}\big(\chi_\alpha \varrho^{\frac{1}{\alpha-\beta}} \geq \lambda\big).$$

Speed of explosion in the critical case $1 > \beta = \alpha > 0$

Theorem

Suppose $R \in \mathcal{R}_\alpha$ *at* ∞ *and* $-\psi \in \mathcal{R}_\alpha$ *at* 0 *for* $\alpha \in (0,1]$, *then*

$$\mathbb{P}_1\big(T_\infty^+ < \infty\big) > 0 \quad \text{if and only if} \quad \int^\infty \frac{-1}{R(y)\psi(1/y)y}\, dy < \infty.$$

In this case, let φ *be defined in* (1), *we have*

$$\frac{T_\infty^+ - T_x^+}{\varphi(x)}\bigg|_{\mathbb{P}_1(\cdot|T_\infty^+<\infty)} \to \Gamma(\alpha) \quad \text{in probability.}$$

Outline of proof

$$m_1(x) \sim \frac{1}{\Gamma(\alpha)} \int_x^\infty \frac{-\mathrm{d}z}{zR(z)\psi(1/z)} = \frac{\varphi(x)}{\Gamma(\alpha)}.$$

$$m_2(x) \sim \frac{2}{\Gamma(\alpha)} \int_x^\infty \frac{-m_1(z)\mathrm{d}z}{zR(z)\psi(1/z)} \sim \frac{-2}{\Gamma^2(\alpha)} \int_x^\infty \varphi'(z)\varphi(z)\mathrm{d}z = \frac{\varphi^2(x)}{\Gamma^2(\alpha)}.$$

Then

$$\left.\frac{T_\infty^+}{\varphi(x)}\right|_{\mathbb{P}_1(\cdot\,|\,T_\infty^+<\infty)} \Rightarrow \frac{1}{\Gamma(\alpha)}$$

Almost surely convergence results

We also obtained some almost sure convergence results for $\frac{T_\infty^+ - T_x^+}{\varphi(x)}$ as $x \to \infty$ for the cases

- $\beta > \alpha = 1$,
- $1 > \beta = \alpha > 0$.

Thank you for your attention!