# On the discrete-time Derrida-Retaux model

Yueyun Hu (Paris XIII)

based on joint works with

- Xinxing Chen, Victor Dagard, Bernard Derrida, Mikhail Lifshits, and Zhan Shi (AoP 2021)
- Xinxing Chen and Zhan Shi (PTRF 2022) and (preprint 2023+)

*"Branching Processes and Applications"*
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# Overview

## Model (Derrida and Retaux) :

Let $X_0$ be random variable taking values in $\{0, 1, 2, ...\}$. Let

$$X_{n+1} \stackrel{\text{law}}{=} (X_n^{(1)} + X_n^{(2)} - 1)^+, \qquad \forall n \geq 0,$$

with two independent copies $X_n^{(1)}$, $X_n^{(2)}$ of $X_n$.

## Question :

What can we say about the asymptotic behaviors of $X_n$ as $n \to \infty$ ?

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Three regimes :

- $X_n \to \infty$ (exponentially fast) : the supercritical regime ;
  - Derrida–Retaux' conjecture on the free energy ;
  - further discussions.
- $X_n \to 0$ (polynomial decay) : the critical regime ;
  - Open questions on the behaviors of $X_n$.
- $X_n \to 0$ (exponential decay) : the subcritical regime ;
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# Outline

# The discrete Derrida–Retaux model

## Motivations/Related models

- Toy model of a hierarchical renormalization model [infinite order phase transition on the localized/delocalized regime] ;
- Collet, Eckmann, Glaser and Martin (1984) [a simplified spin-glass model] ;
- Aldous and Bandyopadhyay (2005) "Max-type recursive distributional equations" ;
- Parking problem on a tree (Goldschmidt and Przykucki (2016), Curien and Hénard (2019), Contat and Curien (2021+)).

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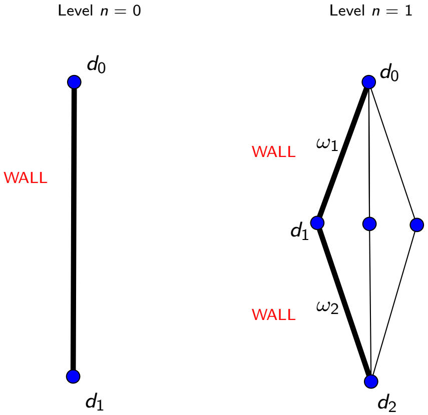
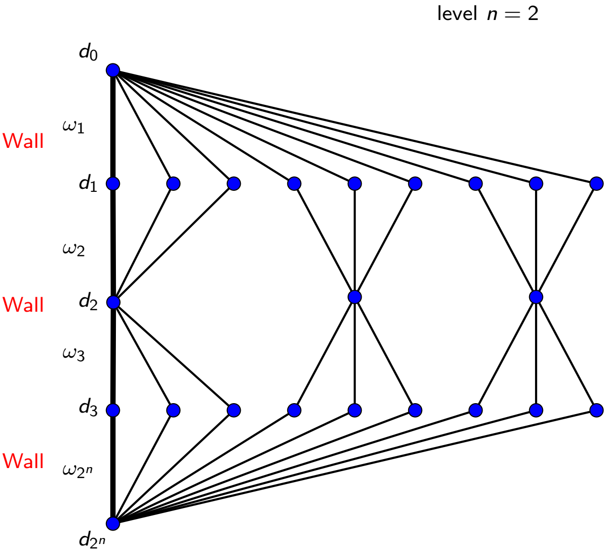
Pinning model on a hierarchical lattice. Derrida, Hakim and Vannimenus (1992) :

Fix an integer $B \geq 2$ (for e.g. $B = 3$).

1. At level 0, there is a unique segment.
2. Rule : Each segment gives $B$ branches consisting of 2 segments each.



The partition function $Z_n := E_S \exp\left(\sum_{i=1}^{2^n} \omega_i 1_{\{S_{i-1}=d_{i-1}, S_i=d_i\}}\right)$.



# Pinning model on a hierarchical lattice

- We have
$$Z_{n+1} = \frac{Z_n^{(1)} Z_n^{(2)} + B - 1}{B},$$
with two independent copies $Z_n^{(1)}$, $Z_n^{(2)}$ of $Z_n$.
- See Monthus and Garet (2008), Derrida, Giacomin, Lacoin and Toninelli (2009), Lacoin and Toninelli (2009), Giacomin, Lacoin and Toninelli (2010, 2011), Berger and Toninelli (2013) for the studies of this model [disorder relevance, critical line...]

# Pinning model on a hierarchical lattice

- Let $X_n := \log Z_n$. Then

$$
\begin{aligned}
X_{n+1} &= \log Z_{n+1} \\
&= \log \frac{e^{(X_n^{(1)}+X_n^{(2)})} + B - 1}{B} \\
&\sim X_n^{(1)} + X_n^{(2)}, \qquad \text{if } X_n^{(1)} + X_n^{(2)} \text{ is large.}
\end{aligned}
$$

- If $X_n \geq \log(B-1)$, a.s., then $X_{n+1} \geq \log(B-1)$ a.s.

# Derrida and Retaux' model

- For any $n \geq 0$,

$$X_{n+1} \stackrel{\text{law}}{=} \max(X_n^{(1)} + X_n^{(2)}, \log(B-1)),$$

with two independent copies $X_n^{(1)}$, $X_n^{(2)}$ of $X_n$.

- After a linear transformation, the recursive equation becomes

$$X_{n+1} \stackrel{\text{law}}{=} (X_n^{(1)} + X_n^{(2)} - 1)^+, \qquad \forall n \geq 0,$$

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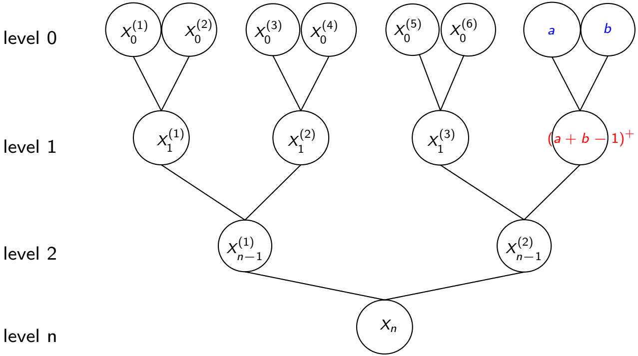

# Derrida and Retaux' model

- Free energy : $F_\infty := \lim_{n\to\infty} \frac{\mathbb{E}(X_n)}{2^n} \in [0,\infty)$ exists.

  *Proof : As* $X_n \overset{\text{law}}{=} (X_{n-1}^{(1)} + X_{n-1}^{(2)} - 1)^+$,
  $2\mathbb{E}(X_{n-1}) \geq \mathbb{E}(X_n) \geq 2\mathbb{E}(X_{n-1}) - 1$, *implying that*

$$F_\infty := \lim_{n\to\infty} \downarrow \frac{\mathbb{E}(X_n)}{2^n} = \lim_{n\to\infty} \uparrow \frac{\mathbb{E}(X_n) - 1}{2^n}.$$

- Let

$$X_0 \overset{\text{law}}{=} (1-p)\delta_{\{0\}} + p\delta_{\{\xi\}},$$

  with $0 \leq p \leq 1$ and $\xi > 0$ a positive random variable. Define

$$p_c := \sup\{0 \leq p \leq 1 : F_\infty(p) = 0\}.$$

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# Derrida and Retaux' model

Question 1 : Value of $p_c$.

- Example : if $X_0 \overset{\text{law}}{=} (1-p)\delta_{\{0\}} + p\delta_{\{2\}}$, then $p_c = ?$

Question 2 :

- If $p_c < 1$, what is the behavior of $F_\infty(p)$ as $p \downarrow p_c$ ? [nearly supercritical regime] ;
- At $p = p_c$, what is the behavior of $X_n$ ? [critical regime] ;
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# Value of $p_c$

**Theorem** (Collet, Eckman, Glaser and Martin 1984)
Let $X_0 \overset{\text{law}}{=} (1-p)\delta_{\{0\}} + p\delta_{\{\xi\}}$. Suppose that $\xi \in \{1, 2, \ldots\}$. Then

$$p_c = \frac{1}{1 + \mathbb{E}((\xi - 1)2^{\xi})}.$$

As example, if $\xi \equiv 2$, then $p_c = \frac{1}{5}$.

**Open problem**
Find $p_c$ for a general r.v. $\xi \in \mathbb{R}_+$ ; or even when $\xi \in \frac{1}{2}\mathbb{N}$ ?

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### Corollary

For any general r.v. $\xi \in \mathbb{R}_+$,

$$p_c > 0 \Longleftrightarrow \mathbb{E}(\xi\, 2^{\xi}) < \infty.$$

### Open question

Is there any *probabilistic* proof (like that of Lyons, Pemantle and Peres) on the above $L \log L$-criterion ?

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# Universalities at or near criticality : the supercritical case

## The Derrida–Retaux conjecture on the free energy

Let $X_0 \stackrel{\text{law}}{=} (1-p)\delta_{\{0\}} + p\delta_{\{\xi\}}$. Suppose that $\xi \in \{1, 2, ...\}$. Under some integrability assumptions on $\xi$,

$$F_\infty(p) = \exp\Big(-\frac{K + o(1)}{(p-p_c)^{1/2}}\Big), \qquad p \downarrow p_c,$$

for some constant $K > 0$.

## Universality at the critical case.

Let $X_0 \overset{\text{law}}{=} (1-p)\delta_{\{0\}} + p\delta_{\{\xi\}}$ and $\xi \in \{1, 2, \ldots\}$.

### Conjecture

If $\mathbb{E}(\xi^3\, 2^\xi) < \infty$ and $p = p_c$, then

- $$\mathbb{P}(X_n \neq 0) \sim \frac{4}{n^2}, \qquad n \to \infty.$$
- Conditionally on $\{X_n \neq 0\}$, $X_n \xrightarrow{(d)} \text{Geometric}(\frac{1}{2})$.

### Conjecture (stable case)

If $\mathbb{P}(\xi = k) \sim c 2^{-k} k^{-\alpha}$ with $2 < \alpha \leq 4$ and $p = p_c$, then

$$\mathbb{P}(X_n \neq 0) \sim \frac{\alpha(\alpha-2)}{2n^2}, \qquad n \to \infty.$$

see Derrida and Shi (2020).

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## The dual Derrida–Retaux conjecture at the subcritical case.

Let $X_0 \overset{\text{law}}{=} (1-p)\delta_{\{0\}} + p\delta_{\{\xi\}}$ and $\xi \in \{1, 2, ...\}$.

### Conjecture

Under suitable integrability assumption on $\xi$, for $p < p_c$,

$$\log \mathbb{P}(X_n \neq 0) \sim -I(p)n, \qquad n \to \infty,$$

with

$$I(p) \sim c(p_c - p)^{1/2}, \qquad p \uparrow p_c.$$

# The supercritical case : results on the free energy

## Main assumption

Let $X_0 \overset{\text{law}}{=} (1-p)\delta_{\{0\}} + p\delta_{\{\xi\}}$. Assume that $\xi$ takes values in $\{1, 2, ...\}$ and

1. either $\mathbb{E}(\xi^3 2^\xi) < \infty$
2. or $\exists\, \alpha \in (-\infty, 4]$ such that $\mathbb{P}(\xi > x) \approx x^{-\alpha} 2^{-x},\ x \to \infty$.

Consequence of Collet, Eckman, Glaser and Martin 1984 :

$p_c > 0$ ($\iff \mathbb{E}[X_0 2^{X_0}] < \infty$) iff $\alpha \in (2, 4]$ or $\mathbb{E}(\xi^3 2^\xi) < \infty$.

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# Nearly supercritical regime, case $p_c > 0$ : Chen, Dagard, Derrida, H., Lifshits, Shi (2021)

Theorem (a weaker version of Derrida–Retaux' conjecture)

If $\mathbb{E}(\xi^3 2^\xi) < \infty$, then

$$F_\infty(p) = \exp\Big( - (p - p_c)^{-\frac{1}{2}+o(1)} \Big), \qquad p \downarrow p_c.$$

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**Theorem**

If $\mathbb{P}(\xi > x) \approx x^{-\alpha} 2^{-x}$ with $2 < \alpha \leq 4$, then

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Remarks :

- If $\mathbb{P}(\xi > x) \sim c\, x^{-2} 2^{-x}$, then $F_\infty(p) = \exp\big( - e^{(c+o(1))/p} \big), p \downarrow 0$.
- See Chen and Shi (2021) for the *stable* case $(2 < \alpha < 4)$.

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# The critical case : Chen, H., Shi (2022)

**Theorem**

If $\mathbb{E}(r^{X_0}) < \infty$ for some $r > 2$ and $p = p_c$, then

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# The subcritical case : Chen, H., and Shi (2023+)

## Theorem

If $\mathbb{E}(r^{X_0}) < \infty$ for some $r > 2$ and $p = p_c - \varepsilon$ with $\varepsilon \in (0, p_c)$, then

$$-c'\varepsilon^{\frac{1}{2}+o(1)} \leq \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{E}(X_n) \leq \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}(X_n) \leq -c\varepsilon^{\frac{1}{2}}.$$

The same holds if we replace $\mathbb{E}(X_n)$ by $\mathbb{P}(X_n \neq 0)$.

## Open problem

Prove, under some suitable integrability assumption on the law of $X_0$, the existence of $\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}(X_n)$ for all $p \in (0, p_c)$.

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# Extension : The Galton-Watson case

Let $\nu$ be an integer-valued r.v. such that $m := \mathbb{E}(\nu) \in (1, \infty)$. Consider the recursive equation

$$X_{n+1} \stackrel{\text{law}}{=} (\sum_{i=1}^{\nu} X_n^{(i)} - 1)^+,$$

where $X_n^{(1)}, X_n^{(2)}, ...,$ are i.i.d. copies of $X_n$, and independent of $\nu$.

## Problem

Study the same questions (critical $p_c$, Derrida–Retaux' conjecture) for this case.

- Only solved when $\nu = m$ equals some integer $m \geq 3$ a.s.
- Case when $X_0$ and $\nu$ belong to a special family of distributions. Work in progress with Gerold Alsmeyer and Bastien Mallein.

## Further open problems and heuristics

- See Derrida and Shi (2020).

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# Proofs of Theorems :

- Analytic tool (for the upper bound) : using the generating functions ;
- Probabilistic tool (for the lower bound) : coupling with the critical Derrida–Retaux tree.

# Analytic tool : generating functions

- Let $G_n(s) := \mathbb{E}(s^{X_n})$. Then

$$G_{n+1}(s) = \frac{1}{s} G_n(s)^2 + (1 - \frac{1}{s}) G_n(0)^2.$$

- Taking the derivative and removing the term $G_n(0)$, we get

$$G_{n+1}(s) - s(s-1) G'_{n+1}(s) = G_n(s)(G_n(s) - 2(s-1) G'_n(s)).$$

- In particular at $s = 2$ we get

$$G_{n+1}(2) - 2 G'_{n+1}(2) = G_n(2)(G_n(2) - 2 G'_n(2)).$$

  See Collet et al. (1984).
- Consequence : $p = p_c$ iff $G_0(2) - 2G'_0(2) = 0$.

# Proofs of Derrida–Retaux conjecture and the dual version

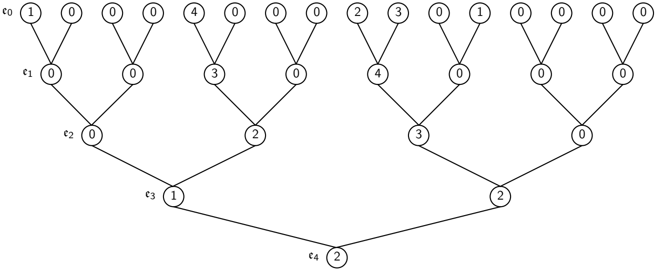
- DR conjecture and the dual version :

$$\begin{aligned} F_\infty(p) &\approx e^{-K(p-p_c)^{-\frac{1}{2}}}, \qquad p \downarrow p_c, \\ \mathbb{P}(X_n \neq 0) &\approx e^{-K'(p_c-p)^{\frac{1}{2}} n}, \qquad n \to \infty,\ p \uparrow p_c. \end{aligned}$$

- Where does it come from the rate $\varepsilon^{\pm\frac{1}{2}}$ with $\varepsilon := |p - p_c|$ ? Heuristically, the system needs a time of order $\varepsilon^{-\frac{1}{2}}$ before drifting away definitely.
- Coupling with the critical Derrida–Retaux tree.

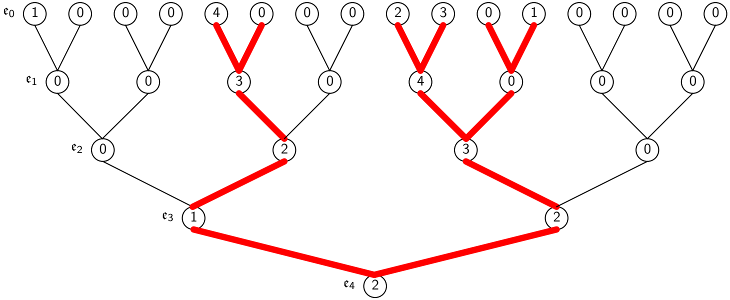
# Probabilistic tool : the DR tree $\mathbb{T}^{(red)}$

- Let $(Y_n)$ be a Derrida-Retaux process in the critical regime (i.e. $Y_0 \overset{\text{law}}{=} (1-p_c)\delta_0 + p_c\delta_\xi$).
- Let $\mathbb{T}$ be an infinite binary tree with $Y(x), |x| = 0$ being i.i.d. copies of $Y_0$.
- Define for any $x \in \mathbb{T}$, $Y(x) := (Y(x^{(1)}) + Y(x^{(2)}) - 1)^+$.
- Let $\mathfrak{e}_n$ be the first lexicographic vertex in the $n$-th generation of the binary tree $\mathbb{T}$ [then $Y_n \overset{\text{law}}{=} Y(\mathfrak{e}_n)$ for any $n \geq 0$].



# Open paths (in red) in $Y(x), x \in \mathbb{T}$

- For any $x \in \mathbb{T}$, we call $(x_k, 0 \le k \le |x|)$ a path leading to $x$ if $x_{k+1}$ is the (unique) child of $x_k$ for any $0 \le k < |x| - 1$.
- A path is said open if for any vertex $x$ in the path, $Y(x^{(1)}) + Y(x^{(2)}) \ge 1$.
- Let $\mathbb{T}_n^{red}$ be the tree of open paths leading to $\mathfrak{e}_n$ and $N_n$ be the number of leaves of $\mathbb{T}_n^{red}$. [Below $N_4 = 6$.]



# Proof in the nearly supercritical regime

To show :
If $\mathbb{E}(X_0^3 2^{X_0}) < \infty$, then

$$F_\infty(p) = \exp\Big( -(p-p_c)^{-\frac{1}{2}+o(1)} \Big), \qquad p \downarrow p_c.$$

Remark
Recall that $F_\infty = \lim_{n\to\infty} \uparrow \frac{\mathbb{E}(X_n)-1}{2^n}$. It is equivalent to show

$$n_0 = (p-p_c)^{-\frac{1}{2}+o(1)},$$

where $n_0 := \inf\{n \geq 1 : \mathbb{E}(X_n) > 1\}$.

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## Coupling between the supercritical regime and the critical regime

- Define $(X_0, Y_0)$ such that $X_0 \stackrel{\text{law}}{=} (1-p)\delta_0 + p\delta_\xi$ with $p = p_c + \varepsilon$ such that $X_0 \geq Y_0$ a.s. and $\mathbb{P}(X_0 = Y_0 | Y_0 > 0) = 1$.
- Define $X(u), u \in \mathbb{T}$ as for $Y(u), u \in \mathbb{T}$.
- Let $N_n^{(0)}$ be the number of open paths $(x_i)_{0\leq i \leq n}$ leading to $\mathfrak{e}_n$ such that $Y(x_0) = 0$ [Below $N_4^{(0)} = 2$].



$X_n$ "$\geq$" $\varepsilon N_n^{(0)}$.

- (Coupling inequality) For all $r \geq 0, n, k \geq 1, \ell \leq \varepsilon r/2$,

$$\mathbb{E}(X_{n+k+\ell}) \geq 2^{\ell-1} \varepsilon \mathbb{E}\left(N_n^{(0)} 2^{Y_n} 1_{\{N_n^{(0)} \geq r, Y_n = k)\}}\right).$$

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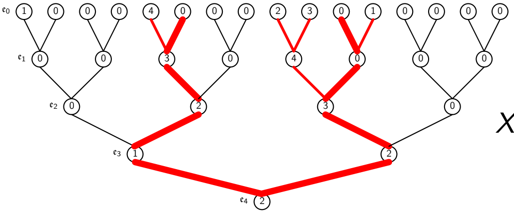


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(lower bound) To show :

If $\mathbb{E}(r^{X_0}) < \infty$ for some $r > 2$ and $\varepsilon > 0$ is small, then

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_n \geq 1) \geq -\varepsilon^{\frac{1}{2}+o(1)}.$$

Lemma

We have

$$\mathbb{P}(X_n \geq 1) \geq \mathbb{E}[(\frac{p}{p_c})^{N_n} 1_{\{Y_n \geq 1\}}].$$

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Conditioned on $\{Y_n \geq 1\}$, $N_n \approx n^2$.

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# Proof in the subcritical case : lower bound.

## Proposition

We have

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(Y_n \geq \frac{n}{j},\ N_n \leq jn) \geq -j^{-1+o(1)}.$$

Consequence :

$$\mathbb{P}(X_n \geq 1) \geq \mathbb{E}[(\frac{p}{p_c})^{N_n} 1_{\{Y_n \geq 1, N_n \leq jn\}}] \geq (\frac{p}{p_c})^{jn} e^{-j^{-1+o(1)} n},$$

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## Conjecture : Derrida and Shi (2020)

Let $x > 0$. Conditionally on $Y_n = \lfloor xn \rfloor$, $\frac{N_n}{n^2}$ converges in law, furthermore, $\frac{1}{n}\mathbb{T}_n^{red}$ converges under the Gromov-Hausdorff metric to a random continuous tree $\mathcal{T}$ [$\mathcal{T}$ appeared in H., Mallein and Pain (2020) as the limit of a continuous-time Derrida–Retaux model in the critical regime].

*THANK YOU!*