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A Mini-Course on

Continuous-State Branching Processes

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0 Introduction

Let $\{\xi_{n,i} : n, i \geq 1\}$ be a family of i.i.d. random variables taking values in $\mathbb{N} := \{0, 1, \dots\}$.

Given the initial state $X(0) \in \mathbb{N}$, one can define a discrete-*state* (*space-time*) branching process $\{X(n) : n \geq 0\}$ by (Bienaymé, 1845; Galton–Watson, 1874)

$$X(n) = \sum_{i=1}^{X(n-1)} \xi_{n,i}, \quad n \geq 1. \tag{0.1}$$

- The one-step transition probabilities satisfy the branching property

$$Q(x + y, \cdot) = Q(x, \cdot) * Q(y, \cdot), \quad x, y \in \mathbb{N}. \tag{0.2}$$

- Consider a sequence of branching processes $\{X_k(n) : n \geq 0\}$, $k \geq 1$.
- A continuous-state branching process $\{x(t) : t \geq 0\}$ arises as the limit (Feller '51)

$$x(t) = \lim_{k\to\infty} \frac{1}{k} X_k(\lfloor kt \rfloor), \qquad t \geq 0. \tag{0.3}$$

A continuous-state branching process $\{x(t) : t \geq 0\}$ is a Markov process in $\mathbb{R}_+ := [0, \infty)$ with transition semigroup $(Q_t)_{t\geq 0}$ satisfying the branching property

$$Q_t(x+y,\cdot) = Q_t(x,\cdot) * Q_t(y,\cdot), \qquad x, y \geq 0. \tag{0.4}$$

This implies $\{Q_t(x,\cdot) : x \geq 0\}$ is a convolution semigroup on $[0, \infty)$, so

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda y} Q_t(x, \mathrm{d}y) = \mathrm{e}^{-x v_t(\lambda)}, \qquad \lambda, t, x \geq 0, \tag{0.5}$$

where $(v_t)_{t\geq 0}$ is the cumulant semigroup with representation

$$v_t(\lambda) = h_t\lambda + \int_0^\infty (1 - \mathrm{e}^{-\lambda y}) l_t(\mathrm{d}y). \tag{0.6}$$

- **Lecture I:** Basic structures and construction of $(v_t)_{t\geq 0}$.

Let $\boldsymbol{Q}_v$ be the law in a suitable path space of the process $\{x(t) : t \geq 0\}$ with $x(0) = v$. Then $(\boldsymbol{Q}_v : v \geq 0)$ is a convolution semigroup.

- **Lecture II:** Lévy-Itô representation of the path-valued Lévy process.

Recall that a discrete-state branching process is defined from i.i.d. random variables $\{\xi_{n,i}\}$ by

$$X(n) = \sum_{i=1}^{X(n-1)} \xi_{n,i}. \tag{0.7}$$

It follows that

$$\begin{aligned} X(n) &= X(n-1) + \sum_{i=1}^{X(n-1)} (\xi_{n,i} - 1), \\ X(n) &= X(0) + \sum_{k=1}^{n} \sum_{i=1}^{X(k-1)} (\xi_{k,i} - 1). \end{aligned}$$

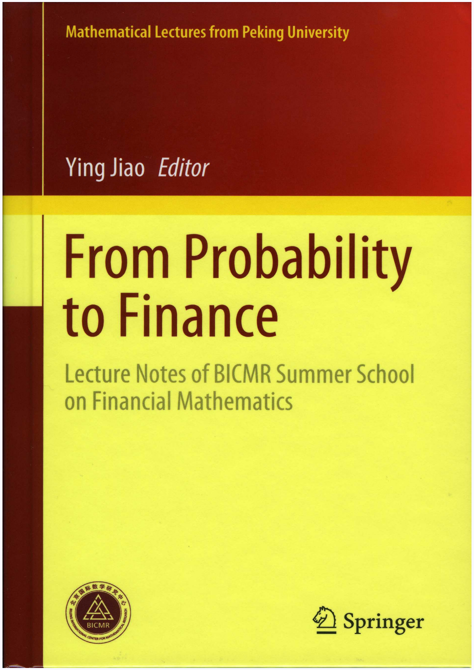
A continuous-state branching process $\{x(t) : t \geq 0\}$ solves (Bertoin–Le Gall '06; Dawson–Li '06/'12)

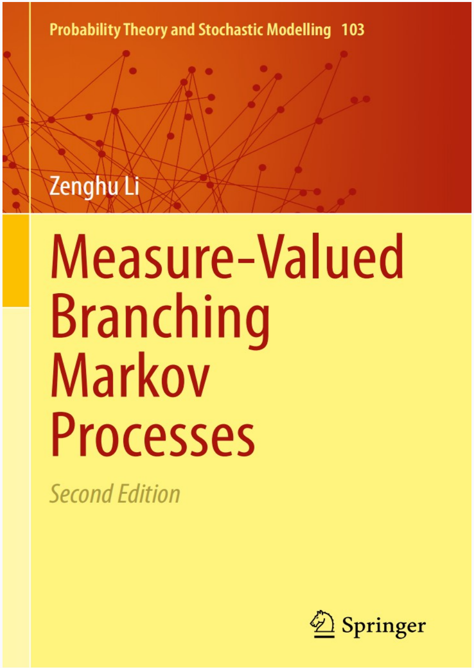
$$x(t) = x(0) + \int_0^t \int_0^{x(s-)} L(\mathrm{d}s, \mathrm{d}u), \tag{0.8}$$

where $L(\mathrm{d}s, \mathrm{d}u)$ is a spectrally positive Lévy white noise on $(0, \infty)^2$.

- **Lecture III:** Existence and uniqueness of the solution to (0.8) and applications.

Sources of the materials





Jiao ('20, Chap. 1) and Li ('22)

Lecture I: The CB-semigroup

1 Branching and immigration structures

1.1 The branching property

A Markov transition semigroup $(Q_t)_{t\geq 0}$ on the state space $[0,\infty)$ is called a continuous-state branching semigroup (CB-semigroup) if it satisfies the branching property:

$$Q_t(x+y,\cdot) = Q_t(x,\cdot) * Q_t(y,\cdot), \quad t,x,y\geq 0, \tag{1.1}$$

where "$*$" denotes the convolution operation.

A set of self-maps $(v_t)_{t\geq 0}$ of $[0,\infty)$ is called a cumulant semigroup if

- (Lévy–Kthintchine representation) for $t\geq 0$ we have

$$v_t(\lambda) = h_t\lambda + \int_{(0,\infty)} (1-\mathrm{e}^{-\lambda y}) l_t(\mathrm{d}y), \quad \lambda\geq 0; \tag{1.2}$$

- (Semigroup property) for $t,r\geq 0$,

$$v_{r+t}(\lambda) = v_r\circ v_t(\lambda) = v_r(v_t(\lambda)), \quad \lambda\geq 0. \tag{1.3}$$

Theorem 1.1 *There is a 1-1 correspondence between CB-semigroups* $(Q_t)_{t\geq 0}$ *and cumulant semigroups* $(v_t)_{t\geq 0}$, *which is given by*

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda y} Q_t(x, \mathrm{d}y) = \mathrm{e}^{-x v_t(\lambda)}, \quad \lambda, t, x \geq 0. \tag{1.4}$$

- A Markov process $\{X(t) : t \geq 0\}$ is called a continuous-state branching process (CB-process) if its transition semigroup is a CB-semigroup.

Theorem 1.2 *The CB-semigroup* $(Q_t)_{t\geq 0}$ *given by* (1.4) *is a Feller semigroup if and only if* $(v_t)_{t\geq 0}$ *is a continuous cumulant semigroup, i.e.* $t \mapsto v_t(\lambda)$ *is continuous for every* $\lambda \geq 0$.

Corollary 1.3 *A CB-process with continuous cumulant semigroup has a realization as a (càdlàg) Hunt process.*

Theorem 1.4 *If* $\{x_1(t) : t \geq 0\}$ *and* $\{x_2(t) : t \geq 0\}$ *are two independent CB-processes with transition semigroup* $(Q_t)_{t\geq 0}$, *then so is* $\{x_1(t) + x_2(t) : t \geq 0\}$.

1.2 Structures of immigration

Suppose that $(Q_t)_{t\geq 0}$ is the CB-semigroup defined by (1.4) from a continuous cumulant semigroup $(v_t)_{t\geq 0}$. Let $(\gamma_t)_{t\geq 0}$ be a family of probability measures on $[0,\infty)$.

We call $(\gamma_t)_{t\geq 0}$ a skew convolution semigroup (SC-semigroup) associated with $(Q_t)_{t\geq 0}$ provided

$$\gamma_{r+t} = (\gamma_r Q_t) * \gamma_t, \qquad r,t \geq 0. \tag{1.5}$$

Theorem 1.5 *The family of probability measures $(\gamma_t)_{t\geq 0}$ on $[0,\infty)$ is an SC-semigroup if and only if a Markov transition semigroup $(Q_t^\gamma)_{t\geq 0}$ on $[0,\infty)$ is defined by*

$$Q_t^\gamma(x,\cdot) = Q_t(x,\cdot) * \gamma_t, \qquad t,x \geq 0. \tag{1.6}$$

If $\{y(t) : t \geq 0\}$ is a Markov process with transition semigroup $(Q_t^\gamma)_{t\geq 0}$ given by (1.6), we call it a continuous-state branching process with immigration (CBI-process) associated with $(Q_t)_{t\geq 0}$.

- By (1.6), the population at time $t \geq 0$ is made up of two parts; the native part generated by the mass $x \geq 0$ has distribution $Q_t(x,\cdot)$ and the immigration part has distribution γ_t.

On the skew convolution equation:

$$\begin{array}{ccccc} (0, r+t] & = & (0, r] & \cup & (r, r+t] \\ \downarrow & & \downarrow & & \downarrow \\ \gamma_{r+t} & & \gamma_r \overset{t}{\rightsquigarrow} \gamma_r Q_t & & \gamma_t \end{array}$$

Theorem 1.6 *The family of probability measures $(\gamma_t)_{t\geq 0}$ on $[0,\infty)$ is an SC-semigroup if and only if its Laplace transform has the representation*

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda y}\gamma_t(\mathrm{d}y) = \exp\Big\{-\int_0^t \psi(v_s(\lambda))\mathrm{d}s\Big\}, \qquad t,\lambda\geq 0, \tag{1.7}$$

where

$$\psi(\lambda) = \beta\lambda + \int_{(0,\infty)} \big(1-\mathrm{e}^{-\lambda z}\big)\nu(\mathrm{d}z). \tag{1.8}$$

- If $v_t(\lambda)\equiv\lambda$, then $(\gamma_t)_{t\geq 0}$ reduces to a classical convolution semigroup:

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda y}\gamma_t(\mathrm{d}y) = \mathrm{e}^{-t\psi(\lambda)}, \qquad t,\lambda\geq 0. \tag{1.9}$$

Let $(Q_t^\gamma)_{t\geq 0}$ be defined by (1.6) with the SC-semigroup $(\gamma_t)_{t\geq 0}$ given by (1.7). Then

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda y} Q_t^\gamma(x, \mathrm{d}y) = \exp\Big\{ -x v_t(\lambda) - \int_0^t \psi(v_s(\lambda))\mathrm{d}s \Big\}. \tag{1.10}$$

We call ψ the immigration mechanism of a CBI-process with transition semigroup $(Q_t^\gamma)_{t\geq 0}$.

Theorem 1.7 *If $(\gamma_1(t))_{t\geq 0}$ and $(\gamma_2(t))_{t\geq 0}$ are two SC-semigroups associated with $(Q_t)_{t\geq 0}$, then so is $(\gamma_1(t) * \gamma_2(t))_{t\geq 0}$.*

Theorem 1.8 *Suppose that $\{y_1(t) : t \geq 0\}$ and $\{y_2(t) : t \geq 0\}$ are two independent CBI-processes associated with $(Q_t)_{t\geq 0}$ having immigration mechanisms ψ_1 and ψ_2, respectively. Then $\{y_1(t) + y_2(t) : t \geq 0\}$ is a CBI-process associated with $(Q_t)_{t\geq 0}$ having immigration mechanism $\psi := \psi_1 + \psi_2$.*

2 Construction of CB-processes

2.1 Discrete-state branching processes

Let $\{p(j) : j \in \mathbb{N}\}$ be a probability distribution on the space of positive integers $\mathbb{N} := \{0, 1, 2, \ldots\}$. It is well-known that $\{p(j) : j \in \mathbb{N}\}$ is uniquely determined by its generating function

$$g(z) := \sum_{j=0}^{\infty} p(j) z^j, \qquad |z| \leq 1.$$

Let $\{\xi_{n,i} : n, i = 1, 2, \ldots\}$ be $\mathbb{N}$-valued i.i.d. random variables with distribution $\{p(j) : j \in \mathbb{N}\}$. Given a random variable $x(0) \in \mathbb{N}$ independent of $\{\xi_{n,i}\}$, we define successively

$$x(n) = \sum_{i=1}^{x(n-1)} \xi_{n,i}, \qquad n = 1, 2, \ldots. \tag{2.1}$$

For $i \in \mathbb{N}$ let $Q(i, \cdot) = p^{*i}(\cdot)$. Then $\{x(n) : n \geq 0\}$ is a Markov chain with transition matrix $Q = (Q(i, j) : i, j \in \mathbb{N})$.

It is easy to see that

$$\sum_{j=0}^{\infty} Q(i,j)z^j = \sum_{j=0}^{\infty} p^{*i}(j)z^j = g(z)^i, \quad i \in \mathbb{N}, |z| \leq 1. \tag{2.2}$$

A Markov chain in $\mathbb{N}$ with transition matrix defined by (2.2) is called a discrete-state branching process (DB-process) with branching distribution given by g.

For any $n \geq 1$ the n-step transition matrix of the DB-process is just the n-fold product matrix $Q^n = (Q^n(i,j) : i,j \in \mathbb{N})$.

Proposition 2.1 *For any $n \geq 1$ and $i \in \mathbb{N}$ we have*

$$\sum_{j=0}^{\infty} Q^n(i,j)z^j = g^{\circ n}(z)^i, \qquad |z| \leq 1, \tag{2.3}$$

where $g^{\circ n}$ is defined by $g^{\circ n}(z) = g \circ g^{\circ(n-1)}(z) = g(g^{\circ(n-1)}(z))$ successively with $g^{\circ 0}(z) =$ z by convention.

2.2 Rescaled DB-processes

Let $\{x_k(n) : n \geq 0\}$ be a DB-process with branching distribution given by the probability generating function g_k, where $k = 1, 2, \ldots$. Let $z_k(n) = k^{-1}x_k(n)$.

Then $\{z_k(n) : n \geq 0\}$ is a Markov chain with state space $E_k := \{0, k^{-1}, 2k^{-1}, \ldots\}$ and n-step transition probability $Q_k^n(x, \mathrm{d}y)$ determined by

$$\int_{E_k} \mathrm{e}^{-\lambda y} Q_k^n(x, \mathrm{d}y) = g_k^{\circ n}(\mathrm{e}^{-\lambda/k})^{kx}, \qquad \lambda \geq 0. \tag{2.4}$$

Let $\gamma_k \to \infty$ increasingly as $k \to \infty$. Let $\lfloor \gamma_k t \rfloor$ denote the integer part of $\gamma_k t$.

Given $z_k(0) = x \in E_k$, for any $t \geq 0$ the random variable

$$z_k(\lfloor \gamma_k t \rfloor) = k^{-1} x_k(\lfloor \gamma_k t \rfloor)$$

has distribution $Q_k^{\lfloor \gamma_k t \rfloor}(x, \cdot)$ on E_k determined by

$$\int_{E_k} \mathrm{e}^{-\lambda y} Q_k^{\lfloor \gamma_k t \rfloor}(x, \mathrm{d}y) = \exp\{-x v_k(t, \lambda)\}, \tag{2.5}$$

where

$$v_k(t,\lambda) = -k \log g_k^{\circ\lfloor \gamma_k t \rfloor}(\mathrm{e}^{-\lambda/k}). \tag{2.6}$$

By (2.6), for $\gamma_k^{-1}(i-1) \le t < \gamma_k^{-1} i$ we have

$$\begin{aligned} v_k(t,\lambda) &= v_k(0,\lambda) + \sum_{j=1}^{\lfloor \gamma_k t \rfloor} [v_k(\gamma_k^{-1} j, \lambda) - v_k(\gamma_k^{-1}(j-1), \lambda)] \\ &= \lambda - k \sum_{j=1}^{\lfloor \gamma_k t \rfloor} [\log g_k^{\circ j}(\mathrm{e}^{-\lambda/k}) - \log g_k^{\circ(j-1)}(\mathrm{e}^{-\lambda/k})] \\ &= \lambda - \gamma_k^{-1} \sum_{j=1}^{\lfloor \gamma_k t \rfloor} \phi_k(-k \log g_k^{\circ(j-1)}(\mathrm{e}^{-\lambda/k})) \\ &= \lambda - \gamma_k^{-1} \sum_{j=1}^{\lfloor \gamma_k t \rfloor} \phi_k(v_k(\gamma_k^{-1}(j-1), \lambda)) \\ &= \lambda - \int_0^{\gamma_k^{-1}\lfloor \gamma_k t \rfloor} \phi_k(v_k(s,\lambda)) \mathrm{d}s, \end{aligned} \tag{2.7}$$

where

$$\phi_k(\lambda) = k\gamma_k \log \left[g_k(\mathrm{e}^{-\lambda/k})\mathrm{e}^{\lambda/k}\right]. \tag{2.8}$$

2.3 The branching mechanism

A convex function ϕ on $[0, \infty)$ is called a branching mechanism if it has the representation

$$\phi(\lambda) = b\lambda + c\lambda^2 + \int_{(0,\infty)} \left(\mathrm{e}^{-\lambda z} - 1 + \lambda z\right) m(\mathrm{d}z), \quad \lambda \geq 0, \tag{2.9}$$

where $c \geq 0$ and b are constants and $(z \wedge z^2)m(\mathrm{d}z)$ is a finite measure on $(0, \infty)$.

Condition 2.2 *The sequence $\{\phi_k\}$ is uniformly Lipschitz on $[0, a]$ for every $a \geq 0$ and there is a function ϕ on $[0, \infty)$ so that $\phi_k(\lambda) \to \phi(\lambda)$ uniformly on $[0, a]$ for every $a \geq 0$ as $k \to \infty$.*

Proposition 2.3 *If Condition 2.2 is satisfied, then ϕ is a branching mechanism with representation (2.9).*

Proposition 2.4 *For every branching mechanism ϕ given by (2.9), there is a sequence $\{\phi_k\}$ in the form of (2.8) satisfying Condition 2.2.*

2.4 The CB-semigroup / process

Theorem 2.5 *Suppose that Condition 2.2 holds. Then for every* $a \geq 0$ *we have* $v_k(t, \lambda) \to$ *some* $v_t(\lambda)$ *uniformly on* $[0, a]^2$ *as* $k \to \infty$ *and the limit function solves*

$$v_t(\lambda) = \lambda - \int_0^t \phi(v_s(\lambda))\mathrm{d}s, \qquad \lambda, t \geq 0. \tag{2.10}$$

Theorem 2.6 *Suppose that* ϕ *is a function given by* (2.9)*. Then for any* $\lambda \geq 0$ *there is a unique positive solution* $t \mapsto v_t(\lambda)$ *to* (2.10) *and* $(v_t)_{t\geq 0}$ *is a cumulant semigroup.*

Corollary 2.7 *Under the assumption of Theorem 2.6, there is a Feller CB-semigroup* $(Q_t)_{t\geq 0}$ *defined by*

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda y} Q_t(x, \mathrm{d}y) = \mathrm{e}^{-x v_t(\lambda)}, \quad \lambda, t, x \geq 0. \tag{2.11}$$

We say a CB-process has branching mechanism ϕ if its transition semigroup $(Q_t)_{t\geq 0}$ is defined by (2.11).

Let W denote the space of positive càdlàg paths on $[0, \infty)$ furnished with the Skorokhod topology; see, e.g., Ethier and Kurtz (1986).

Theorem 2.8 *Suppose that Condition 2.2 holds. Let* $\{x(t) : t \geq 0\}$ *be a càdlàg CB-process with transition semigroup* $(Q_t)_{t\geq 0}$ *defined by* (2.10) *and* (2.11). *If* $z_k(0)$ *converges to* $x(0)$ *in distribution, then* $\{z_k(\lfloor \gamma_k t \rfloor) : t \geq 0\}$ *converges to* $\{x(t) : t \geq 0\}$ *in distribution on* W.

Observation: Propositions 2.3 and 2.4 indicate that the CB-processes give exactly all possible scaling limits of DB-processes.

Problems:

- Characterize the class of all possible scaling limits of discrete-state branching processes in i.i.d. random environments.
- Characterize the class of continuous-state branching processes in varying environments defined by

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda y} Q_{r,t}(x, \mathrm{d}y) = \mathrm{e}^{-x v_{r,t}(\lambda)}, \quad \lambda, x \geq 0, t \geq r \geq 0; \tag{2.12}$$

  see Bansaye and Simatos (2015, EJP) and Fang and L (2022, AOAP).

Lecture II: The Lévy–Itô representation

3 Some simple properties

3.1 Reviews

The branching mechanism ϕ of a CB-process is a convex function on $[0, \infty)$ given by

$$\phi(\lambda) = b\lambda + c\lambda^2 + \int_{(0,\infty)} \big(\mathrm{e}^{-\lambda z} - 1 + \lambda z\big) m(\mathrm{d}z), \quad \lambda \geq 0, \tag{3.1}$$

where $c \geq 0$ and b are constants and $(z \wedge z^2) m(\mathrm{d}z)$ is a finite measure on $(0, \infty)$. We have

$$\phi'(\lambda) = b + 2c\lambda + \int_{(0,\infty)} z\big(1 - \mathrm{e}^{-\lambda z}\big) m(\mathrm{d}z), \qquad \lambda \geq 0. \tag{3.2}$$

For this branching mechanism, there is a CB-semigroup $(Q_t)_{t\geq 0}$ such that

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda y} Q_t(x, \mathrm{d}y) = \mathrm{e}^{-x v_t(\lambda)}, \quad \lambda, t, x \geq 0, \tag{3.3}$$

where $t \mapsto v_t(\lambda)$ is the unique positive solution of

$$v_t(\lambda) = \lambda - \int_0^t \phi(v_s(\lambda)) \mathrm{d}s, \qquad \lambda, t \geq 0. \tag{3.4}$$

The CB-process could be constructed for more general ϕ (including Neveu's $\phi(\lambda) \equiv \lambda \log \lambda$).

3.2 Forward and backward differential equations

From (3.4) we see that $t \mapsto v_t(\lambda)$ is first continuous and then continuously differentiable. Moreover, it is easy to see that

$$\frac{\partial}{\partial t} v_t(\lambda)\Big|_{t=0} = -\phi(\lambda), \qquad \lambda \geq 0.$$

By the semigroup property $v_{t+s} = v_s \circ v_t = v_t \circ v_s$, we get the backward differential equation:

$$\frac{\partial}{\partial t} v_t(\lambda) = -\phi(v_t(\lambda)), \qquad v_0(\lambda) = \lambda, \tag{3.5}$$

and forward differential equation:

$$\frac{\partial}{\partial t} v_t(\lambda) = -\phi(\lambda)\frac{\partial}{\partial \lambda} v_t(\lambda), \quad v_0(\lambda) = \lambda. \tag{3.6}$$

We can also rewrite (3.6) into its integral form:

$$v_t(\lambda) = \lambda - \int_0^t \phi(\lambda)\frac{\partial}{\partial \lambda} v_s(\lambda)\mathrm{d}s, \quad t \geq 0. \tag{3.7}$$

3.3 The first moment

From (3.3) and (3.4) it follows that

$$\int_{[0,\infty)} y Q_t(x, \mathrm{d}y) = x\mathrm{e}^{-\phi'(0)t} = x\mathrm{e}^{-bt}, \qquad t \geq 0, x \geq 0. \tag{3.8}$$

Proposition 3.1 *If $\{x(t) : t \geq 0\}$ is a CB-process with branching mechanism ϕ, then $\{\mathrm{e}^{bt}x(t) : t \geq 0\}$ is a positive martingale.*

3.4 The spectrally positive Lévy process

The branching mechanism ϕ is the Laplace exponent of a spectrally positive Lévy process, which is connected with the CB-process by Lamperti's time changes; see Kyprianou (2014, Springer).

We have ($0 \cdot \infty = 0$)

$$\phi'(0) = b, \quad \phi'(\infty) = b + 2c \cdot \infty + \int_{(0,\infty)} z m(\mathrm{d}z).$$

The Lévy process has infinite variations if and only it $\phi'(\infty) = \infty$.

3.5 The extinction time

By Corollary 1.3, the CB-process has a Hunt process realization $X = (\Omega, \mathscr{F}, \mathscr{F}_t, x(t), \boldsymbol{Q}_x)$. Let $\tau_0 := \inf\{s \geq 0 : x(s) = 0\}$ denote the *extinction time*.

Theorem 3.2 *For every $t \geq 0$ the limit $\bar{v}_t = \lim_{\lambda\to\infty} v_t(\lambda)$ exists in $(0, \infty]$. Moreover, the mapping $t \mapsto \bar{v}_t$ is decreasing and for any $t \geq 0$ and $x > 0$ we have*

$$\boldsymbol{Q}_x\{\tau_0 \leq t\} = \boldsymbol{Q}_x\{x(t) = 0\} = \exp\{-x\bar{v}_t\}. \tag{3.9}$$

Condition 3.3 *(Grey's condition) There is some constant $\theta > 0$ so that*

$$\phi(z) > 0 \text{ for } z \geq \theta \text{ and } \int_\theta^\infty \phi(z)^{-1}\mathrm{d}z < \infty.$$

Theorem 3.4 *We have $\bar{v}_t < \infty$ for some and hence all $t > 0$ if and only if Condition 3.3 holds. In this case, $t \mapsto \bar{v}_t = l_t(0, \infty)$ is the unique solution to*

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{v}_t = -\phi(\bar{v}_t), \qquad \bar{v}_{0+} = \infty. \tag{3.10}$$

3.6 The canonical entrance rule

The cumulant semigroup $(v_t)_{t\geq 0}$ has the canonical Lévy–Kthintchine representation:

$$v_t(\lambda) = h_t\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda z}) l_t(\mathrm{d}z), \qquad t \geq 0, \lambda \geq 0, \tag{3.11}$$

Write $Q_t^\circ(x, \mathrm{d}z) = 1_{\{z>0\}} Q_t(x, \mathrm{d}z)$. Then $v_{r+t} = v_r \circ v_t$ implies, for all $t, r > 0$,

$$h_{r+t} = h_r h_t, \ \ l_{r+t}(\mathrm{d}z) = h_r l_t(\mathrm{d}z) + l_r Q_t^\circ(\mathrm{d}z), \tag{3.12}$$

and so $l_{r+t} \leq l_r Q_t^\circ$. We call $(l_t)_{t>0}$ the canonical entrance rule.

Theorem 3.5 *We have $h_t = 0$ for some and hence all $t > 0$ if and only if $\phi'(\infty) = \infty$. In this case, the family $(l_t)_{t>0}$ is an entrance law, i.e., $l_{r+t} = l_r Q_t^\circ$ for all $t, r > 0$.*

- In the situation of Theorem 3.5, as $x \downarrow 0$,

$$\int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda z}) x^{-1} Q_t^\circ(x, \mathrm{d}z) = x^{-1}(1 - \mathrm{e}^{-x v_t(\lambda)}) \uparrow v_t(\lambda) = \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda z}) l_t(\mathrm{d}z).$$

3.7 The space of paths

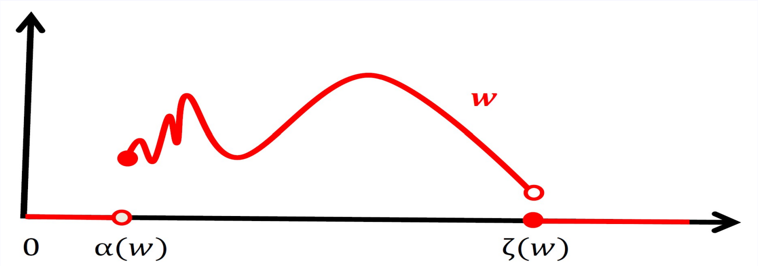
Let W be the space of positive càdlàg paths on $[0, \infty)$ furnished with the σ-algebras

$$\mathscr{W} = \sigma(\{w(s) : 0 \leq s < \infty\}), \;\; \mathscr{W}_t = \sigma(\{w(s) : 0 \leq s \leq t\}), \quad t \geq 0.$$

For $w \in W$ let $\alpha(w) = \inf\{s \geq 0 : w(s) > 0\}$ and

$$\zeta(w) = \inf\{s > \alpha(w) : w(s) \text{ or } w(s-) = 0\}.$$

Let $\hat{W} = \{w \in W : w(t) = 0 \text{ for } t < \alpha(w) \text{ and } t \geq \zeta(w)\} \subset W$.



4 Canonical Kuznetsov measures

4.1 The canonical excursion law

Theorem 4.1 *Suppose that $\phi'(\infty) = \infty$ and let $(l_t)_{t>0}$ be the canonical entrance law. Then there is a unique σ-finite measure $\boldsymbol{N}_0$ on $(W, \mathscr{W})$ supported by*

$$W_0 := \{w \in \hat{W} : \alpha(w) = w(0) = 0\} \subset \hat{W}$$

such that, for $0 < t_1 < t_2 < \cdots < t_n$ and $x_1, x_2, \ldots, x_n > 0$,

$$\begin{aligned} &\boldsymbol{N}_0(w(t_1) \in \mathrm{d}x_1, w(t_2) \in \mathrm{d}x_2, \ldots, w(t_n) \in \mathrm{d}x_n) \\ &\quad = l_{t_1}(\mathrm{d}x_1) Q^\circ_{t_2-t_1}(x_1, \mathrm{d}x_2) \cdots Q^\circ_{t_n-t_{n-1}}(x_{n-1}, \mathrm{d}x_n). \end{aligned} \tag{4.1}$$

- Roughly speaking, (4.1) means $\{w(t) : t \geq 0\}$ under $\boldsymbol{N}_0$ is a CB-process.
- Let $(W, \mathscr{W}, \mathscr{W}_t, w(t), \boldsymbol{Q}_x)$ be the canonical càdlàg realization of the CB-process. Formally,

$$l_t = \lim_{x\to 0} x^{-1} Q^\circ_t(x, \cdot) \;\; \Rightarrow \;\; \boldsymbol{N}_0 = \lim_{x\to 0} x^{-1} \boldsymbol{Q}_x \;\; \Rightarrow \;\; \mathrm{supp}(\boldsymbol{N}_0) \subset W_0. \tag{4.2}$$

- The rigorous proof of the above theorem depends on the following Proposition 4.2.

Proposition 4.2 *Let $\phi_0'(\lambda) = \phi'(\lambda) - b$ for $\lambda \geq 0$, where ϕ' is given by (3.2). We can define a Feller transition semigroup $(Q_t^b)_{t\geq 0}$ on $[0,\infty)$ by*

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda y} Q_t^b(x, \mathrm{d}y) = \exp\Big\{ - x v_t(\lambda) - \int_0^t \phi_0'(v_s(\lambda))\mathrm{d}s \Big\}. \tag{4.3}$$

Moreover, we have

$$Q_t^b(x, \mathrm{d}y) = \mathrm{e}^{bt} x^{-1} y Q_t(x, \mathrm{d}y), \quad x > 0 \tag{4.4}$$

and

$$Q_t^b(0, \mathrm{d}y) = \mathrm{e}^{bt}[h_t \delta_0(\mathrm{d}y) + y l_t(\mathrm{d}y)]. \tag{4.5}$$

- The transition semigroup $(Q_t^b)_{t\geq 0}$ defined by (4.3) is that of a special CBI-process.
- Let $(W, \mathscr{W}, \mathscr{W}_t, w(t), \boldsymbol{Q}_x^b)$ be the canonical càdlàg realization of the CBI-process. Then

$$w(T)\boldsymbol{N}_0(\mathrm{d}w) = \mathrm{e}^{-bT}\boldsymbol{Q}_0^b(\mathrm{d}w) \text{ on } \mathscr{W}_T = \sigma(\{w(s) : 0 \leq s \leq T\}), \tag{4.6}$$

  which gives rigorously $\mathrm{supp}(\boldsymbol{N}_0) \subset W_0$.

4.2 The canonical Kuznetsov measure

Theorem 4.3 *When* $\delta := \phi'(\infty) < \infty$*, we have* $c = 0$ *and, for* $t \geq 0$ *and* $\lambda \geq 0$*,*

$$v_t(\lambda) = \mathrm{e}^{-\delta t}\lambda + \int_0^t \mathrm{e}^{-\delta s}\mathrm{d}s \int_{(0,\infty)} (1 - \mathrm{e}^{-z v_{t-s}(\lambda)}) m(\mathrm{d}z). \tag{4.7}$$

Consequently, the Lévy–Kthintchine formula (3.11) *for* $v_t(\lambda)$ *holds with*

$$h_t = \mathrm{e}^{-\delta t}, \quad l_t = \int_0^t \mathrm{e}^{-\delta s} m Q^\circ_{t-s} \mathrm{d}s, \quad t \geq 0. \tag{4.8}$$

Theorem 4.4 *Suppose that* $\delta := \phi'(\infty) < \infty$*. Then there is a* σ*-finite measure* $\boldsymbol{N}_1$ *carried by*

$$W_1 := \{w \in \hat{W} : \alpha(w) > 0, w(\alpha(w)) > 0\} \subset \hat{W}$$

such that, for any $t > r \geq 0$*,* $F \in b\mathscr{W}_r$ *and* $\lambda \geq 0$*,*

$$\begin{aligned} \boldsymbol{N}_1[F(w)(1 - \mathrm{e}^{-\lambda w(t)})] \;=\; & \boldsymbol{N}_1[F(w)(1 - \mathrm{e}^{-v_{t-r}(\lambda) w(r)})] \\ & + F([0])[\mathrm{e}^{-\delta r} v_{t-r}(\lambda) - \mathrm{e}^{-\delta t}\lambda]. \end{aligned} \tag{4.9}$$

- The proof of Theorem 4.4 is based on the relations in (4.8).
- In Markov processes, $\boldsymbol{N}_1$ is known as a Kuznetsov measure; see, e.g., Dellacherie et al. (1992).

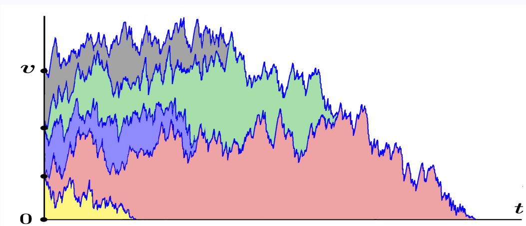
5 Structures of sample paths

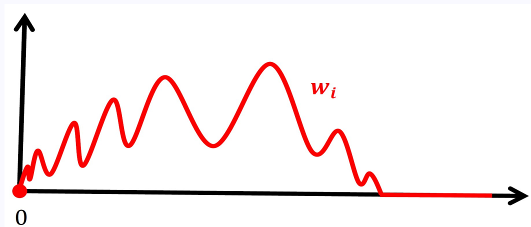
5.1 Cluster representations of the CB-process

Theorem 5.1 *Suppose that $\phi'(\infty) = \infty$. Let $v \geq 0$ and let $N(\mathrm{d}w) = \sum_{i=1}^{\infty} \delta_{w_i}(\mathrm{d}w)$ be a Poisson random measure on W with intensity $v\mathbf{N}_0(\mathrm{d}w)$, where $\mathbf{N}_0$ is the excursion law (carried by $W_0 \subset \hat{W}$). Let $X_0 = v$ and for $t > 0$ let*

$$X_t = \int_W w(t) N(\mathrm{d}w) = \sum_{i=1}^{\infty} w_i(t). \tag{5.1}$$

Then $\{X_t : t \geq 0\}$ is a realization of the CB-process.

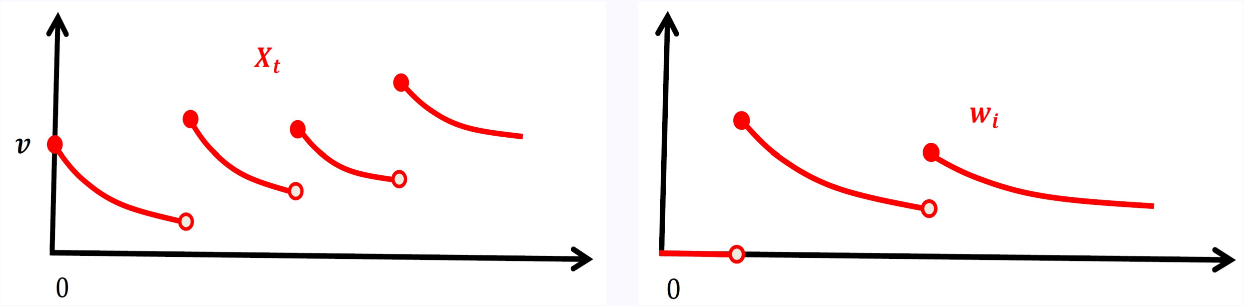




Theorem 5.2 *Suppose that* $\delta := \phi'(\infty) < \infty$. *Let* $v \geq 0$ *and let* $N(\mathrm{d}w) = \sum_{i=1}^{\infty} \delta_{w_i}(\mathrm{d}w)$ *be a Poisson random measure on* W *with intensity* $v\mathbf{N}_1(\mathrm{d}w)$, *where* $\mathbf{N}_1$ *is the canonical Kuznetsov measure (carried by* $W_1 \subset \hat{W}$*). For any* $t \geq 0$ *let*

$$X_t = v\mathrm{e}^{-\delta t} + \int_W w(t) N(\mathrm{d}w) = v\mathrm{e}^{-\delta t} + \sum_{i=1}^{\infty} w_i(t). \tag{5.2}$$

Then $\{X_t : t \geq 0\}$ *is a realization of the CB-process.*



5.2 Flow of CB-processes

Let $X = (W, \mathscr{W}, \mathscr{W}_t, w(t), \boldsymbol{Q}_x)$ be a canonical realization of the CB-process. By Theorem 1.4,

$$\boldsymbol{Q}_{v_1+v_2} = \boldsymbol{Q}_{v_1} * \boldsymbol{Q}_{v_2}, \quad v_1, v_2 \geq 0. \tag{5.3}$$

Then $(\boldsymbol{Q}_v)_{v\geq 0}$ is a convolution semigroup on the path space $(W, \mathscr{W})$.

Proposition 5.3 *Suppose that $\phi'(\infty) = \infty$. Let $N(\mathrm{d}w, \mathrm{d}u)$ be a Poisson random measure on $W \times (0, \infty)$ with intensity $\mathbf{N}_0(\mathrm{d}w)\mathrm{d}u$. For $v \geq 0$ let $X_0(v) = v$ and*

$$X_t(v) = \int_W \int_0^v w(t) N(\mathrm{d}u, \mathrm{d}w), \quad t > 0. \tag{5.4}$$

Then $\{X_t(v) : t \geq 0\}$ is a realization of the CB-process.

Proposition 5.4 *Suppose that $\delta := \phi'(\infty) < \infty$. Let $N(\mathrm{d}w, \mathrm{d}u)$ be a Poisson random measure on $W \times (0, \infty)$ with intensity $\mathbf{N}_1(\mathrm{d}w)\mathrm{d}u$. For $v, t \geq 0$ let*

$$X_t(v) = v\mathrm{e}^{-\delta t} + \int_W \int_0^v w(t) N(\mathrm{d}w, \mathrm{d}u). \tag{5.5}$$

Then $\{X_t(v) : t \geq 0\}$ is a realization of the CB-process.

Let ρ be the metric on W defined by ($b = \phi'(0)$)

$$\rho(w_1, w_2) = \sup_{s\geq 0} \mathrm{e}^{bs}|w_1(s) - w_2(s)|, \quad w_1, w_2 \in W. \tag{5.6}$$

Theorem 5.5 *There is a version of the random field $\{X_t(v) : v \geq 0, t \geq 0\}$ defined by (5.4) or (5.5) with the following properties:*

(i) *The path-valued process $\{X(v) : v \geq 0\}$ is increasing and ρ-càdlàg and has stationary and independent increments.*

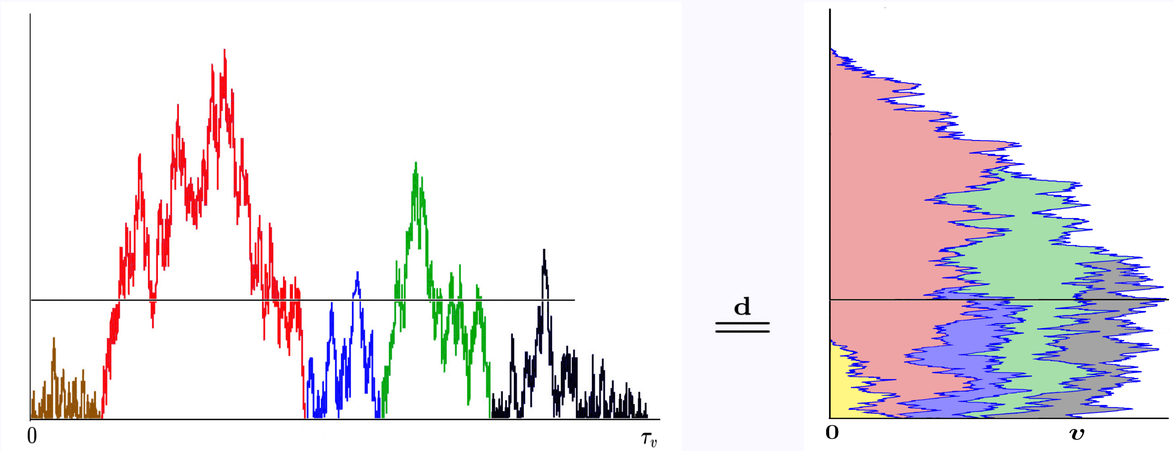
(ii) *For any $v_2 \geq v_1 \geq 0$ the difference $X(v_2) - X(v_1) = \{X_t(v_2) - X_t(v_2) : t \geq 0\}$ is a CB-process with transition semigroup $(Q_t)_{t\geq 0}$.*

Remarks:

- The path-valued process $\{X(v) : v \geq 0\}$ is a Lévy process with state space $(W, \mathscr{W})$, and (5.4) and (5.5) give its Lévy–Itô representation in the cases $\phi'(\infty) = \infty$ and $< \infty$, respectively.
- The random field $\{X_t(v) : v \geq 0, t \geq 0\}$ is a realization of the flow of subordinators introduced by Bertoin and Le Gall (2000, 2003, 2005, 2006).

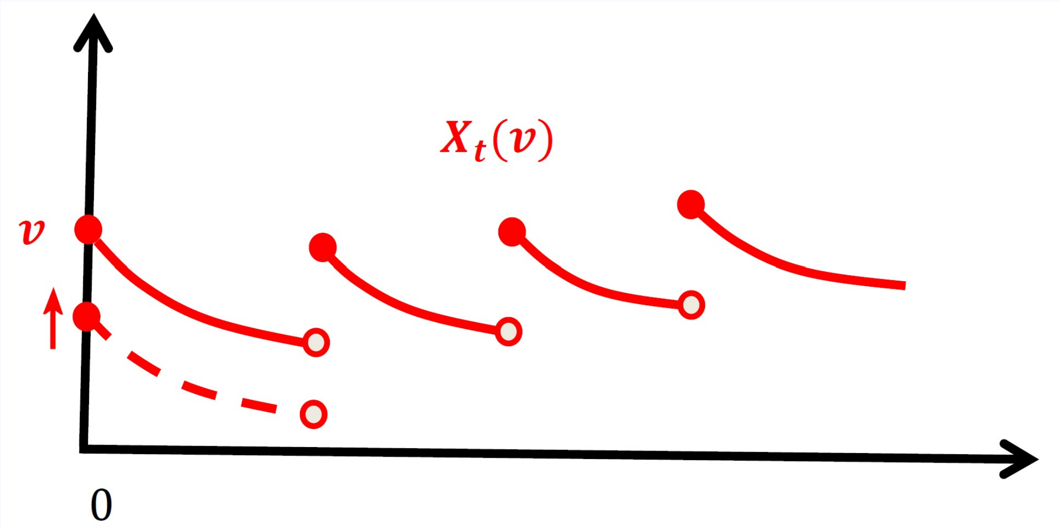
Observations:

- If $\delta := \phi'(\infty) = \infty$, then $v \mapsto \boldsymbol{X}(v) = (\boldsymbol{X}_t(v))_{t\geq 0}$ is pure jump process.



A pure jump increasing path-valued Lévy process $v \mapsto \boldsymbol{X}(v) = (\boldsymbol{X}_t(v))_{t\geq 0}$.

- If $\delta := \phi'(\infty) < \infty$, then $v \mapsto \boldsymbol{X}(v) = (X_t(v))_{t\geq 0}$ has the continuous drift part $v \mapsto (v\mathrm{e}^{-\delta t})_{t\geq 0}$.



Lecture III: Stochastic equations

6 Martingale problems for CBI-processes

6.1 Reviews

The branching and immigration mechanisms (ϕ, ϕ) are functions on $[0, \infty)$ given by

$$\phi(\lambda) = b\lambda + c\lambda^2 + \int_{(0,\infty)} \big(e^{-\lambda z} - 1 + \lambda z\big) m(dz), \tag{6.1}$$

and

$$\psi(\lambda) = \beta\lambda + \int_{(0,\infty)} \big(1 - e^{-\lambda z}\big)\nu(dz), \quad \lambda \geq 0. \tag{6.2}$$

A CBI-process has transition semigroup $(Q_t^\gamma)_{t\geq 0}$ such that

$$\int_{[0,\infty)} e^{-\lambda y} Q_t^\gamma(x, dy) = \exp\Big\{ -xv_t(\lambda) - \int_0^t \psi(v_s(\lambda))ds \Big\}. \tag{6.3}$$

where $t \mapsto v_t(\lambda)$ is the unique positive solution of

$$\frac{\partial}{\partial t} v_t(\lambda) = -\phi(v_t(\lambda)), \qquad v_0(\lambda) = \lambda. \tag{6.4}$$

6.2 A forward integral equation

Proposition 6.1 *For any $t \geq 0$ and $\lambda \geq 0$ we have*

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda y} Q_t^\gamma(x, \mathrm{d}y) = \mathrm{e}^{-x\lambda} + \int_0^t \mathrm{d}s \int_{[0,\infty)} [y\phi(\lambda) - \psi(\lambda)] \mathrm{e}^{-y\lambda} Q_s^\gamma(x, \mathrm{d}y). \tag{6.5}$$

6.3 Equivalent martingale problems

Let $C^{1,2}([0,\infty)^2)$ be the set of bounded continuous real functions $(t, x) \mapsto G(t, x)$ on $[0,\infty)^2$ with bounded continuous derivatives up to the first order relative to $t \geq 0$ and up to the second order relative to $x \geq 0$.

Let $C^2[0,\infty)$ denote the set of bounded continuous real functions on $[0,\infty)$ with bounded continuous derivatives up to the second order. For $f \in C^2[0,\infty)$ define

$$\begin{aligned} Lf(x) \; = \; & cxf''(x) + x \int_{(0,\infty)} \big[f(x+z) - f(x) - zf'(x)\big] m(\mathrm{d}z) \\ & + (\beta - bx)f'(x) + \int_{(0,\infty)} \big[f(x+z) - f(x)\big] \nu(\mathrm{d}z). \end{aligned} \tag{6.6}$$

We shall identify L as the generator of the CBI-process.

Suppose that $(\Omega, \mathscr{G}, \mathscr{G}_t, \boldsymbol{P})$ is a filtered probability space satisfying the usual hypotheses and $\{y(t) : t \geq 0\}$ is a càdlàg process in $[0, \infty)$ that is adapted to $(\mathscr{G}_t)_{t\geq 0}$ and satisfies $\boldsymbol{P}[y(0)] < \infty$. Let us consider the following martingale problems:

(1) For every $T \geq 0$ and $\lambda \geq 0$,

$$\exp\Big\{ - v_{T-t}(\lambda)y(t) - \int_0^{T-t} \psi(v_s(\lambda))\mathrm{d}s\Big\}, \qquad 0 \leq t \leq T,$$

is a martingale.

(2) For every $\lambda \geq 0$,

$$H_t(\lambda) := \exp\Big\{ - \lambda y(t) + \int_0^t [\psi(\lambda) - y(s)\phi(\lambda)]\mathrm{d}s\Big\}, \quad t \geq 0,$$

is a local martingale.

(3) The process $\{y(t) : t \geq 0\}$ a semi-martingale with no negative jumps and the optional random measure

$$N_0(\mathrm{d}s, \mathrm{d}z) := \sum_{s>0} 1_{\{\Delta y(s)\neq 0\}}\delta_{(s,\Delta y(s))}(\mathrm{d}s, \mathrm{d}z),$$

where $\Delta y(s) = y(s) - y(s-)$, has predictable compensator $\hat{N}_0(\mathrm{d}s, \mathrm{d}z) = \mathrm{d}s\nu(\mathrm{d}z) + y(s-)\mathrm{d}sm(\mathrm{d}z)$. Let $\tilde{N}_0(\mathrm{d}s, \mathrm{d}z) = N_0(\mathrm{d}s, \mathrm{d}z) - \hat{N}_0(\mathrm{d}s, \mathrm{d}z)$. We have

$$y(t) = y(0) + M^c(t) + M^d(t) - b\int_0^t y(s-)\mathrm{d}s + \psi'(0)t,$$

where $\{M^c(t) : t \geq 0\}$ is a continuous local martingale with quadratic variation $2cy(t-)\mathrm{d}t$ and

$$M^d(t) = \int_0^t \int_{(0,\infty)} z\tilde{N}_0(\mathrm{d}s, \mathrm{d}z), \qquad t \geq 0,$$

is a purely discontinuous local martingale.

(4) For every $f \in C^2[0, \infty)$ we have

$$f(y(t)) = f(y(0)) + \int_0^t Lf(y(s))\mathrm{d}s + \text{local mart.} \tag{6.7}$$

(5) For any $G \in C^{1,2}([0, \infty)^2)$ we have

$$G(t, y(t)) = G(0, y(0)) + \int_0^t \left[G_t'(s, y(s)) + LG(s, y(s))\right]\mathrm{d}s + \text{local mart.} \tag{6.8}$$

where L acts on the function $x \mapsto G(s, x)$.

Theorem 6.2 *The above properties (1), (2), (3), (4) and (5) are equivalent to each other. Those properties hold if and only if* $\{(y(t), \mathscr{G}_t) : t \geq 0\}$ *is a CBI-process with branching mechanism* ϕ *and immigration mechanism* ψ*.*

Corollary 6.3 *Let* $\{(y(t), \mathscr{G}_t) : t \geq 0\}$ *be a càdlàg realization of the CBI-process satisfying* $\boldsymbol{P}[y(0)] < \infty$*. Then the above properties (3), (4) and (5) hold with the local martingales being martingales.*

7 Stochastic equations for CBI-processes

In this and the next section, we understand

$$\int_a^b = \int_{(a,b]} \text{ and } \int_a^\infty = \int_{(a,\infty)}, \qquad b \geq a \geq 0.$$

7.1 Weak solutions

Let $\{B(t)\}$ be a standard Brownian motion and $\{M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\}$ a Poisson time-space random measure on $(0, \infty)^3$ with intensity $\mathrm{d}s m(\mathrm{d}z)\mathrm{d}u$. Let $\{\eta(t)\}$ be an increasing Lévy process with $\eta(0) = 0$ and with Laplace exponent

$$\psi(\lambda) = -\log \boldsymbol{P} \exp\{-\lambda\eta(1)\}, \quad \lambda \geq 0. \tag{7.1}$$

We assume all those are independent of each other. Consider the stochastic integral equation

$$\begin{aligned} y(t) &= y(0) + \int_0^t \sqrt{2cy(s-)}\mathrm{d}B(s) - b\int_0^t y(s-)\mathrm{d}s \\ &\quad + \int_0^t \int_0^\infty \int_0^{y(s-)} z\tilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \eta(t), \end{aligned} \tag{7.2}$$

where $\tilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) = M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) - \mathrm{d}s m(\mathrm{d}z)\mathrm{d}u$ denotes the compensated measure.

We understand the forth term on the right-hand side of (7.2) as an integral over the random set $\{(s, z, u) : 0 < s \leq t, 0 < z < \infty, 0 < u \leq y(s-)\}$. Similar interpretations are given for other stochastic integral equations like (7.2).

By saying that $\{y(t) : t \geq 0\}$ is a weak solution to (7.2), we mean it is a positive càdlàg process defined on some probability space with the noises $\{B(t)\}$, $\{M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\}$ and $\{\eta(t)\}$ such that the equation holds almost surely for every $t \geq 0$.

We refer to Ikeda and Watanabe (1989) and Situ (2005) for the basic theory of stochastic equations.

Theorem 7.1 *A positive càdlàg process $\{y(t) : t \geq 0\}$ is a CBI-process with branching and immigration mechanisms (ϕ, ψ) given respectively by (2.9) and (1.8) if and only if it is a weak solution to (7.2).*

Let $\{M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\}$ and $\{\eta(s)\}$ be as in (7.2). Let $\{W(\mathrm{d}s, \mathrm{d}u)\}$ be a Gaussian time-space white noise on $(0, \infty)^2$ with intensity $2c\mathrm{d}s\mathrm{d}u$. We assume the noises are independent of each other. Consider the stochastic integral equation

$$\begin{aligned} y(t) &= y(0) + \int_0^t \int_0^{y(s-)} W(\mathrm{d}s, \mathrm{d}u) - b \int_0^t y(s-)\mathrm{d}s \\ &\quad + \int_0^t \int_0^\infty \int_0^{y(s-)} z\tilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \eta(t). \end{aligned} \tag{7.3}$$

Theorem 7.2 *A positive càdlàg process $\{y(t) : t \geq 0\}$ is a CBI-process with branching and immigration mechanisms (ϕ, ψ) given respectively by (2.9) and (1.8) if and only if it is a weak solution to (7.3).*

From (7.2) or (7.3) we see that the immigration of the CBI-process $\{y(t)\}$ is represented by the increasing Lévy process $\{\eta(t)\}$. By the Lévy–Itô decomposition, there is a Poisson time-space random measure $\{N(\mathrm{d}s, \mathrm{d}z)\}$ with intensity $\mathrm{d}s\nu(\mathrm{d}z)$ such that

$$\eta(t) = \beta t + \int_0^t \int_0^\infty zN(\mathrm{d}s, \mathrm{d}z), \qquad t \geq 0.$$

Then the immigration of $\{y(t)\}$ involves two parts: the continuous part determined by the drift coefficient β and the discontinuous part given by the Poisson random measure $\{N(\mathrm{d}s, \mathrm{d}z)\}$.

7.2 Strong solutions and comparisons

Theorem 7.3 *For any initial value* $y(0) = x \geq 0$*, there are pathwise unique positive (strong) solutions to* (7.2) *and* (7.3)*.*

Theorem 7.4 *Suppose that* $\{y_1(t) : t \geq 0\}$ *and* $\{y_2(t) : t \geq 0\}$ *are two positive solutions to* (7.3) *with* $\boldsymbol{P}\{y_1(0) \leq y_2(0)\} = 1$*. Then we have* $\boldsymbol{P}\{y_1(t) \leq y_2(t)$ *for all* $t \geq 0\} = 1$*.*

A comparison properties of the solutions to (7.2) can also be established.

7.3 The time-space white Lévy noise

Let $\{L(\mathrm{d}s, \mathrm{d}u)\}$ be the spectrally positive time-space $(\mathscr{G}_t)$-Lévy white noise on $(0, \infty)^2$ defined by

$$L(\mathrm{d}s, \mathrm{d}u) = W(\mathrm{d}s, \mathrm{d}u) - b\mathrm{d}s\mathrm{d}u + \int_{\{0<z<\infty\}} z\tilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \qquad (7.4)$$

We may rewrite (7.3) as

$$y(t) = y(0) + \int_0^t \int_0^{y(s-)} L(\mathrm{d}s, \mathrm{d}u) + \eta(t), \quad t \geq 0. \tag{7.5}$$

7.4 Flow of CBI-processes

Let $\{L(\mathrm{d}s, \mathrm{d}u)\}$ be the Lévy time-space white noise on $(0, \infty)^2$ defined by (7.4). Let $\{\eta(t)\}$ be an increasing Lévy process with $\eta(0) = 0$ and with Laplace exponent given by (7.1). We assume the noises are independent of each other.

By Theorem 7.3, for each $v \geq 0$ there is a pathwise unique solution $\{Y_t(v) : t \geq 0\}$ to

$$Y_t(v) = v + \int_0^t \int_0^{Y_{s-}(v)} L(\mathrm{d}s, \mathrm{d}u) + \eta(t). \tag{7.6}$$

Recall that W denotes the space of positive càdlàg paths on $[0, \infty)$. Define the metric ρ by

$$\rho(w_1, w_2) = \sup_{s \geq 0} \mathrm{e}^{bs} |w_1(s) - w_2(s)|, \quad w_1, w_2 \in W. \tag{7.7}$$

Theorem 7.5 *There is a version of the random field* $\{Y_t(v) : v \geq 0, t \geq 0\}$ *defined by* (7.6) *with the following properties:*

- *The path-valued process* $\{Y(v) : v \geq 0\}$ *is increasing and* ρ*-càdlàg and has stationary and independent increments.*
- *For any* $v_2 \geq v_1 \geq 0$ *the difference* $Y(v_2) - Y(v_1) = \{Y_t(v_2) - Y_t(v_2) : t \geq 0\}$ *is a CB-process with transition semigroup* $(Q_t)_{t\geq 0}$.

By Theorem 7.5, the path-valued process $\{Y(v) : v \geq 0\}$ is a Lévy process with state space $(W, \mathscr{W})$. The initial state of $\{Y(v) : v \geq 0\}$ is the CBI-process $Y(0) = \{Y_t(0) : t \geq 0\}$.

7.5 Stable Lévy noises

Let $c, q \geq 0$, $b \in \mathbb{R}$ and $1 < \alpha < 2$ be given constants. Let $\{B(t)\}$ be a standard Brownian motion. Let $\{z(t)\}$ be a spectrally positive α-stable Lévy process with Lévy measure

$$\gamma(\mathrm{d}z) := (\alpha - 1)\Gamma(2 - \alpha)^{-1} z^{-1-\alpha} \mathrm{d}z, \qquad z > 0$$

and $\{\eta(t)\}$ an increasing Lévy process with $\eta(0) = 0$ and with Laplace exponent ψ. We assume the noises are independent of each other. Consider the stochastic differential equation

$$\mathrm{d}y(t) = \sqrt{2cy(t-)}\mathrm{d}B(t) + \sqrt[\alpha]{\alpha q y(t-)}\mathrm{d}z(t) - by(t-)\mathrm{d}t + \mathrm{d}\eta(t), \quad (7.8)$$

Theorem 7.6 *A positive càdlàg process $\{y(t) : t \geq 0\}$ is a CBI-process with branching mechanism $\phi(\lambda) = b\lambda + c\lambda^2 + q\lambda^\alpha$ and immigration mechanism ψ given by (1.8) if and only if it is a weak solution to (7.8).*

Theorem 7.7 *For any initial value $y(0) = x \geq 0$, there is a pathwise unique positive strong solution to (7.8).*

7.6 Examples

Suppose that $\{\xi_{n,i} : n, i = 1, 2, \ldots\}$ and $\{\eta_n : n = 1, 2, \ldots\}$ are two independent families of $\mathbb{N}$-valued i.i.d. random variables. Given the initial state $Y(0) \in \mathbb{N}$ independent of $\{\xi_{n,i}\}$ and

$\{\eta_n\}$, we can define a discrete-state branching process with immigration by

$$Y(n) = \sum_{i=1}^{Y(n-1)} \xi_{n,i} + \eta_n, \qquad n \geq 1. \tag{7.9}$$

Example 7.8 *The equation (7.3) can be thought as a continuous time-space counterpart of the definition (7.9) of the DBI-process. In fact, assuming $\mu = \mathbf{E}(\xi_{1,1}) < \infty$, we have*

$$Y(n) = Y(n-1) + \sum_{i=1}^{Y(n-1)} (\xi_{n,i} - \mu) - (1-\mu)Y(n-1) + \eta_n. \tag{7.10}$$

It follows that

$$Y(n) = Y(0) + \sum_{k=1}^{n} \sum_{i=1}^{Y(k-1)} (\xi_{k,i} - \mu) - (1-\mu) \sum_{k=1}^{n} Y(k-1) + \sum_{k=1}^{n} \eta_k. \tag{7.11}$$

The exact continuous time-space counterpart of (7.11) would be the stochastic integral equation

$$y(t) = y(0) + \int_0^t \int_0^\infty \int_0^{y(s-)} \xi \tilde{M}(\mathrm{d}s, \mathrm{d}\xi, \mathrm{d}u) - \int_0^t b y(s-) \mathrm{d}s + \eta(t), \tag{7.12}$$

which is a typical special form of (7.3); see Bertoin and Le Gall (2006) and Dawson and Li (2006).

Example 7.9 *The stochastic differential equation* (7.8) *captures the structure of the CBI-process in a typical special case. Let* $1 < \alpha \leq 2$*. Under the condition* $\mu := \boldsymbol{E}(\xi_{1,1}) < \infty$*, we have*

$$Y(n) - Y(n-1) = \sqrt[\alpha]{Y(n-1)} \sum_{i=1}^{Y(n-1)} \frac{\xi_{n,i} - \mu}{\sqrt[\alpha]{Y(n-1)}} - (1-\mu)Y(n-1) + \eta_n.$$

A continuous time-state counterpart of the above equation would be

$$\mathrm{d}y(t) = \sqrt[\alpha]{\alpha q y(t-)}\mathrm{d}z(t) - by(t)\mathrm{d}t + \beta\mathrm{d}t, \qquad t \geq 0, \tag{7.13}$$

where $\{z(t) : t \geq 0\}$ *is a standard Brownian motion if* $\alpha = 2$ *and a spectrally positive* α*-stable Lévy process. This is a typical special form of* (7.8)*; see Fu and Li (2010).*

Example 7.10 *When* $\alpha = 2$ *and* $\beta = 0$*, the solution to* (7.13) *is a diffusion process and known as Feller's branching diffusion. This process was first studied by Feller (1951).*

8 Recent topics and applications

8.1 Distributional properties of jumps

Let $\{x(t) : t \geq 0\}$ be a CB-process. For $A \in \mathscr{B}(0, \infty)$ let

$$\begin{aligned} x_A(t) &= \text{Card}\{s \in (0,t] : x(s) - x(s-) \in A\}, \\ \tau_A &= \inf\{s > 0 : x(s) - x(s-) \in A\}, \\ M(t) &= \max\{x(s) - x(s-) : s \in (0,t]\}. \end{aligned}$$

Characterizations of the distributions of those random variables can be derived easily from the stochastic equations, say,

$$\begin{aligned} x(t) = x(0) + \int_0^t \int_0^\infty \int_0^{x(s-)} z \tilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \\ x_A(t) = 0 + \int_0^t \int_A \int_0^{y(s-)} M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \end{aligned}$$

The equations show that $\{(x(t), x_A(t)) : t \geq 0\}$ is a two-dimensional CB-process.

8.2 Variation of the transition probabilities

Let $x \geq y \geq 0$ and let $\{x(t) : t \geq 0\}$ and $y(t) : t \geq 0\}$ be CBI-processes defined by

$$x(t) = x + \int_0^t \int_0^{x(s-)} L(\mathrm{d}s, \mathrm{d}u) + \eta(t),$$
$$y(t) = y + \int_0^t \int_0^{x(s-)} L(\mathrm{d}s, \mathrm{d}u) + \eta(t).$$

Then $\{\xi(t) := x(t) - y(t) : t \geq 0\}$ is a CB-process since

$$\xi(t) = x - y + \int_0^t \int_0^{\xi(s-)} L(\mathrm{d}s, x(s-) + \mathrm{d}u).$$

This leads to the useful estimate for the variation of the transition probabilities:

$$\begin{aligned} \|Q_t(x,\cdot) - Q_t(y,\cdot)\|_{\text{var}} &= \sup_{\|f\|\leq 1} \left| \int_{[0,\infty)} f(z) Q_t(x, \mathrm{d}z) - \int_{[0,\infty)} f(z) Q_t(y, \mathrm{d}z) \right| \\ &= \sup_{\|f\|\leq 1} \left| \boldsymbol{E}[f((x(t))] - \boldsymbol{E}[f((y(t))] \right| \\ &\leq \sup_{\|f\|\leq 1} \boldsymbol{E}\big[\left| f((x(t)) - f((y(t)) \right| \big] \leq 2\boldsymbol{P}(\xi(t) \neq 0). \end{aligned}$$

thank you!