The continuum Derrida–Retaux system

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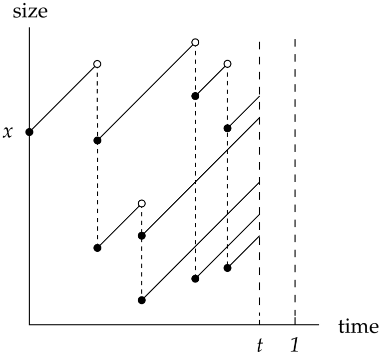
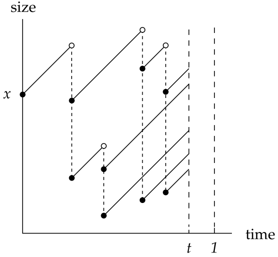


- ▷ At time $t = 0$, one cell with size $x \in [0, \infty)$;
- ▷ Size of each cell increases at constant speed 1;
- ▷ Each cell splits at rate $\frac{2m}{(1-t)^2}$ ($m =$ size, $t =$ time);
- ▷ Splitting: $m \Rightarrow (mU, m(1-U))$ (U= (0, 1)-uniform).

◇ Scaling limit of the discrete Derrida–Retaux model at criticality (talk by Yueyun Hu, a few minutes ago).

◇ A growth-fragmentation process in the sense of Bertoin (2017).



$N(t)$ = number of cells at time t;

$X_1(t), \ldots, X_{N(t)}(t)$: sizes of cells;

$M(t) = X_1(t) + \cdots + X_{N(t)}(t)$.

Hu, Mallein and Pain (2020).

$\exists c_1, c_2 \in (0, \infty)$: $\forall x \ge 0$, when $t \to 1^-$,

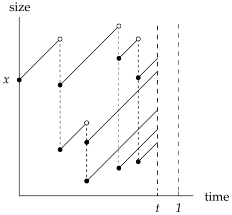
$$(1-t)M(t) \to c_1\xi, \quad (1-t)^2 N(t) \to c_2\xi, \quad \text{(jointly) in law,}$$

$\xi := \int_0^1 \varrho(s)^2 \, \mathrm{d}s$, $(\varrho(s), s \in [0, 1])$ 4-D Bessel bridge, $\varrho(0) = 2\sqrt{x}$, $\varrho(1) = 0$.

Theorem 1. $\forall x \ge 0$, when $t \to 1^-$,

$$(1-t)M(t) \to \eta_\infty, \quad (1-t)^2 N(t) \to \eta_\infty, \quad \mathbb{P}_x\text{-a.s.}$$

η_∞ distributed as $\frac{1}{2}\int_0^1 \varrho(s)^2 \, \mathrm{d}s$.



Theorem 1. $\forall x \geq 0$, when $t \to 1^-$, $\mathbb{P}_x$-a.s.,

$$(1-t)\sum_{i=1}^{N(t)} X_i(t) \to \eta_\infty, \quad (1-t)^2 N(t) \to \eta_\infty.$$

Theorem 2. $\forall x \geq 0$, when $t \to 1^-$, $\mathbb{P}_x$-a.s.,

$$(1-t)^2 \sum_{i=1}^{N(t)} \delta_{\frac{X_i(t)}{1-t}} \to \eta_\infty \gamma \quad \text{weakly.}$$

$$\gamma(\mathrm{d}y) = 4y\, \mathrm{e}^{-2y}\, \mathbf{1}_{\{y>0\}} \mathrm{d}y.$$

Corollary. $\forall x \geq 0$, $\mathbb{P}_x$-a.s., $\frac{1}{N(t)} \sum_{i=1}^{N(t)} \delta_{\frac{X_i(t)}{1-t}} \to \gamma$ weakly.

Bertoin and Watson (2020). Law of large numbers for growth-fragmentation processes under general assumptions.

Technique: $\forall x \geq 0$, $\forall t \in [0, 1)$,

$$\mathbb{E}_x\Big[\mathrm{e}^{\sum_{i=1}^{N(t)} f(t, X_i(t))}\Big] = \mathrm{e}^{f(0,x)} + \int_0^t \mathbb{E}_x\Big[\mathrm{e}^{\sum_{i=1}^{N(r)} f(r, X_i(r))} \sum_{j=1}^{N(r)} (\mathcal{L}f)(r, X_j(r))\Big]\mathrm{d}r\,.$$

$\mathcal{L}$: explicit *nonlinear* operator.

Application: $\mathcal{L}f = 0 \;\Rightarrow\; \mathbb{E}_x[\mathrm{e}^{\sum_{k=1}^{N(t)} f(t, X_i(t))}] = \mathrm{e}^{f(0,\,x)}$.

Possible extensions:

- ◇ More general branching mechanisms.
- ◇ Overlaps.
- ◇ Other interesting quantities
  (example: number of branching events along each branch).